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ABSTRACT

GRADES OR AGES: Grades K-7. SUBJECT MATTER: Mathematics. ORGANIZATION AND PHYSICAL APPEARANCE: The main section of the guide entitled "Mathematical Strands with Teaching Suggestions" has the following subsections: 1) sets and numbers, 2) numeration, 3) operations on whole numbers, 4) rational numbers, 5) geometry, and 6) measurement. Other chapters deal with problem solving and program objectives. The guide is printed and edition bound with a soft cover. OBJECTIVES AND ACTIVITIES: Objectives are listed for each grade under the six subsection headings. Activities are described in detail in the main section of the guide. INSTRUCTIONAL MATERIALS: None are listed. STUDENT ASSESSMENT: No special provision is made for evaluation. (MBM)

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Elementary Mathematics Guide

K-7

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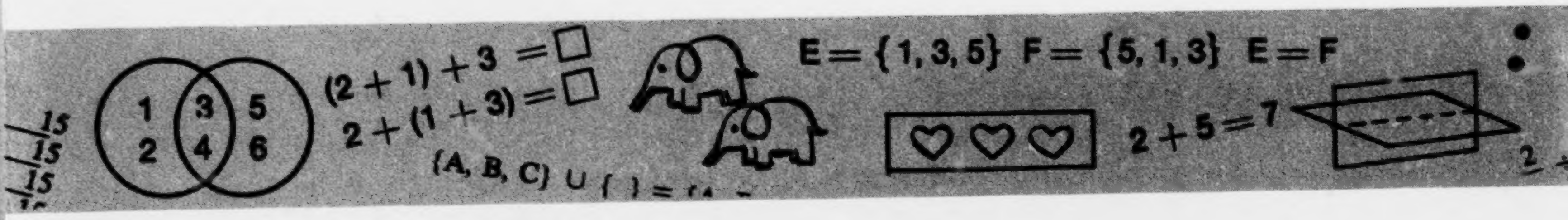

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INTRODUCTION

The Elementary Mathematics Guide—K-7 was prepared to assist elementary teachers in planning and directing a logical, sequential program in modern mathematics. Attention is invited to the first section of the Guide which sets forth reasons for the development of a new guide in elementary mathematics.

The Guide outlines basic ideas used in elementary mathematics and includes teaching suggestions for developing understanding of these ideas. The Statement of Objectives lists ideas and skills which most children might be expected to acquire at a given grade level. The form in which these objectives are presented was designed to give a picture of sequential development of mathematical ideas and skills throughout the elementary school and to indicate scope of emphasis at each grade level.

Intuitive understanding of mathematical ideas which children already possess is used as the basis for developing greater competence in the use of numbers to communicate ideas and to solve problems. Physical models are used to introduce new ideas and to provide concrete means for solving number problems.

Appreciation is expressed to a committee of classroom teachers, elementary principals and supervisors, college personnel, and staff members of the Elementary Education Service of the State Department of Education, which worked over a period of three years to prepare the Guide. Appreciation is also expressed to other teachers, principals, and supervisors who joined members of the committee in summer workshops devoted to the preparation of material incorporated in the Guide.

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Mathematics for the Elementary School

POINT OF VIEW

A scientific and technological society demands more and better training in mathematics for its citizens. The needs in such a society and the changes in mathematics itself require changes in the teaching of mathematics at all levels. Curriculum changes in the elementary school have not, however, been as drastic as popular opinion concerning the "new math" might lead one to expect. A sound curriculum in modern mathematics for the elementary school is distinguished by two major characteristics:

> The mathematical content of the program and
> An approach to teaching that is consistent with the best that is known about learning.

MATHEMATICAL CONTENT

Traditionally the curriculum in the elementary school was called arithmetic since it was centered around computation with whole numbers and fractional numbers. Now we speak of *mathematics* in the elementary school because it has become apparent that the content may be made more meaningful and better continuity provided by the introduction of new topics.

The major new content for elementary school mathematics consists of sets and set language, properties of the various sets of numbers, number sentences, geometry, and a variety of numeration systems. Much of this new content has as its main purpose the improvement of understanding of the usual content of arithmetic.

The use of sets and set language helps to clarify and make more precise the concept of number. The union of disjoint sets of objects provides a model for the addition of numbers. Separating a set of physical objects provides a concrete model for subtraction of numbers.

The ideas and language of sets provide the vehicle for the study of geometric ideas. All geometric figures may be viewed as sets of points. A line is a set of points; a sphere is a set of points; a circle is a set of points. Through the study of sets of points and relations among them children can describe with precision and clarity the shape of objects found in their environment.

The use of sets and set language enables teachers to focus attention on the various number systems that are studied in elementary school—the set of natural or counting numbers, the set of whole numbers, the set of fractional numbers and the set of rational numbers. A study of the unique properties of these various sets of numbers and their applications helps the child see the value of each.

Introduction and use of the properties of the various number systems give meaning to the learning of number facts and computational algorithms. The closure, commutative, associative, distributive, identity and inverse properties are basic laws which provide the predominant structure for number systems used by elementary children.

For example, to find the sum of 7 and 8, the child may think: "I need 3 of the 8 to put with 7 to make 10; $7 + 3 = 10$; and 5 more is 15." The associative property is the basic reason for changing the $7 + (3 + 5)$ to $(7 + 3) + 5$ to make the addition fact easier.

To find the product of 7 and 8 a child may think: "7 eights is $(5 + 2)$ eights. This is $(5 \times 8) + (2 \times 8) = 40 + 16$ or 56." Naming $(5 + 2) \times 8$ as $(5 \times 8) + (2 \times 8)$ utilizes the distributive property.

To find the product of 3 and $4\frac{1}{2}$, the distributive property is used to name

$$3 \times (4 + \frac{1}{2}) \text{ as } (3 \times 4) + (3 \times \frac{1}{2}) =$$

$$12 + 1\frac{1}{2} \text{ or } 13\frac{1}{2}$$

When we think of structure in work with numbers in the elementary school, we think of properties of the number systems (closure, commutative, associative, identity, inverse and distributive) and characteristics of the numeration system (base, digits, place value, and position). Together these provide the basic reasons or main ideas around which a learner organizes his thinking and from which he deduces facts and computational algorithms. Heavy emphasis on the properties does not imply that elementary teachers begin with structure. Instead, by repeated use of the properties, the structure emerges in the child's mind. Many experiences, ranging from the intuitive and concrete to the abstract, in using these key ideas build for the learner a sense of their importance and real value.

Use of number sentences, including equations and inequalities, helps in solving word problems. A useful first step in solving a problem is translation of the physical situation into the mathematical language of a number sentence. The language of mathematics contains symbols comparable to the nouns, verbs and phrases used in other communication. A situation which requires the equal apportionment of three dozen cookies among eighteen children, when translated into a number sentence, may become:

$$(3 \times 12) \div 18 = \square \quad \text{or} \quad N = \frac{(3 \times 12)}{18}$$

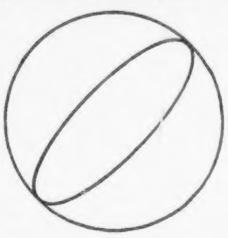
Solving the problem then becomes a matter of completing familiar operations.

The use of both equations and inequalities also helps in estimating computations. The cost of three and one-half dozen eggs at fifty cents per dozen may be quickly estimated by the following inequalities:

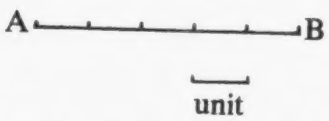
$$3\frac{1}{2} \times 50 \quad > \quad 3 \times 50 \text{ or } \$1.50$$

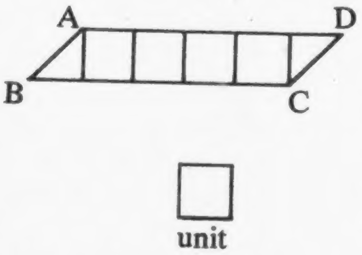
$$3\frac{1}{2} \times 50 \quad < \quad 4 \times 50 \text{ or } \$2.00$$

Geometry in the elementary school is the study of positions and locations in space. A position or location is viewed as a point. Space is the set of all points. Through the study of geometry greater understanding of shapes in the physical environment is obtained. For example: A soap bubble is a physical model for a set of points called a sphere. The intersection of a sphere and a plane passing through the center of the sphere is a great circle. The great circle is the route for the shortest distance between any two points on the earth's surface.

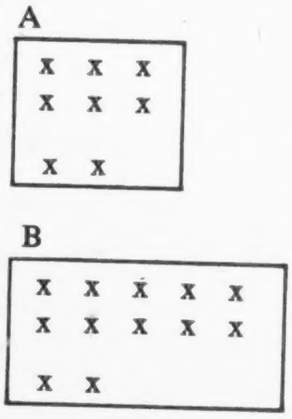


Geometry is necessary for precise and correct development of concepts of measurement. Length, for example, is a number of units obtained by comparing a given line segment with a chosen unit. The length of $\overline{AB}$ is 5 units, using the unit shown. Area is a number of units obtained by comparing a unit region to a given region. The area of parallelogram region ABCD is 5, using the square region shown as the unit. By stressing the association of numbers with geometric objects in measurement and through the use of a number line, a helpful model is provided for whole numbers, fractional numbers and operations on them.





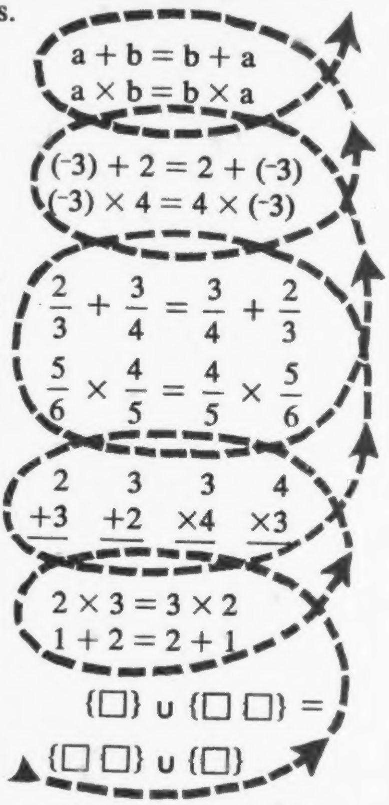
Through the study of a variety of numeration systems, children are helped to broaden their view of essential characteristics of numeration systems and to appreciate the rich historical background of mathematics. Naming the number of members in set A as 12_{six}—one set of six and 2 more—gives contrast to the base ten in the numeration system which we use, when 12_{ten} names the number of members in set B. Naming the number 4 in Roman numerals as IV shows the use of subtraction in a numeration system which contrasts with the use of only addition in the decimal place value system. The introduction of other numeration systems thus broadens and deepens a fundamental mathematical idea and provides one avenue for building interests and appreciations.



The new content described above is believed to make mathematics more meaningful for the learner. The new content also enables the elementary school to provide for better continuity as students study more and more mathematics. However, the new content does not constitute the major portion of the elementary school mathematics curriculum. The dominant emphasis will continue, as in the past, to be the study of whole numbers and fractional numbers.

Spiral Development of Mathematical Content

With the goal to provide ever-expanding views of central mathematical ideas, content is *spiraled* from grade to grade. Ideas are introduced at a given grade, redeveloped and deepened in subsequent grades. For example, the commutative, associative and distributive properties for addition and multiplication of whole numbers are developed informally and experimentally in the early grades. They are later named and used



COMMUTATIVE PROPERTY

as basic reasons for algorithms. Then as the set of fractional numbers is studied, properties for addition and multiplication are viewed again for application to fractional numbers. Likewise, when the set of integers, the set of rational numbers and the set of real numbers are studied, properties for addition and multiplication are again verified. With each successive extension of the number system, the goal is to ascertain which properties fit operations on the new numbers and which properties are new for the extended number systems.

A numeration system with base of ten is introduced in grade one by grouping objects into sets of ten. Place value is then introduced. The basic notation of base and place value are spiraled throughout the curriculum with the goal of deepening the level of generality and abstraction. The use of expanded notation, expanded notation with exponents, Roman numerals, other ancient numeration systems, and other bases with place value are examples of the spiral nature of numeration. As long as a person continues the study of mathematics, patterns will be generalized and abstracted at more sophisticated levels.

Symbols and Vocabulary

The introduction of new topics into the content of elementary school mathematics has made necessary the use of *new symbols* and *new vocabulary*. While vocabulary and symbols are useful in the communication of ideas in mathematics, the introduction of new terminology is second to the development of understanding of the designated idea. For example, the names of properties of number systems—closure, commutative, associative, identity, inverse and distributive—should be delayed until the student has a reasonable degree of understanding of the ideas. There should be opportunity to talk about the idea, to use it, and then to learn its name for oral and written use. This procedure is equally true for symbols—the use of braces to denote a set, and the symbols denoting union and separation; for the inequalities symbols denoting "is greater than" and "is less than"; and for such new vocabulary in geometry as line segment, closed path, plane region and space region.

While there is a slight increase in the total number of technical words and symbols in general use, certain words that have been in the program have been deleted. Some words deleted, such as "minuend," "subtrahend," "multiplicand" and "multiplier," have doubtful value for the learner and are not maintained in subsequent mathematics courses. Other deletions such as "borrowing" and "carrying" have been made because they tend to convey misleading ideas.

Terminology is an integral part of the content of elementary school mathematics, but if introduced before understanding is developed, it may handicap or inhibit development of ideas. By the age of nine or ten many children are interested and have readiness for new terminology. However, caution should be exercised to insure that words and symbols do not become confused with mathematical content.

APPROACH TO TEACHING

Development of Understanding. One main goal of modern mathematics in schools is the development of greater understanding on the part of pupils. Greater understanding means longer retention, better application of new ideas, and improved power in reasoning and problem solving. This has been a goal for mathematics education for many years and much of the present movement is toward the further realization and delineation of this goal.

Development of Competence in Computation. Concomitant with the goal of understanding is the development of competence in computation. Subsequent mathematical ideas are more difficult, if not impossible, to develop without skill in computation. The skill to be expected by the teacher must vary with individual pupils. Knowledge of pupil maturity, and the mathematical or social need for the skill determine expectations set by the teacher.

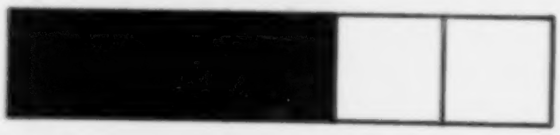
Experiences with Concrete Materials. In developing understanding of mathematics and competence in computation it is important that a child's first experience with any new topic be related to his previous background and experience. For many new topics pupils need to handle and manipulate physical objects to provide experience needed to make the transition to abstract ideas. For example, experience which the child has in physically moving sets of objects together provides foundation for developing the concept of addition. Much such physical combining is needed for the learner to make a smooth transition to imagined combining of objects and finally to the abstraction. In the primary grades where formal work in making the transition from concrete objects to abstract ideas begins, it is important that each child have manipulative materials at hand which he is free to use at any time to help in the discovery of a number fact or the verification of a mathematical idea.

Transition from Concrete to Abstract. As pupils' concepts become more mature, the need for experience with concrete materials decreases because they are able to use previous ideas to learn new ones. However, the transition from concrete to abstract should not be too abrupt. Many children may need to work from time to time with sets of objects or physical models in order to re-establish understanding or to visualize a new concept. The teacher should encourage interplay between concrete and abstract for purposes of application, redevelopment and review.

Involvement of Pupils in Instructional Goals. It is generally agreed that those goals which children accept as their own are most effective in motivating learning. Helping children set worthwhile goals is difficult. They more readily accept instructional goals when there is interest in the topic, when the topic is related to familiar ideas, when application of knowledge and skill can be made to their own lives or when a challenge is accepted. Having set a goal, children must have opportunity to gain a sense of achievement. The teacher should guarantee this sense of achievement for each pupil who exerts a reasonable amount of effort.

Involvement Through Discovery. It is the goal for every child in the class to become actively involved in the learning process. An atmosphere of discovery helps achieve this goal. In a discovery oriented class, teachers ask thought provoking questions, wait for responses from children and build subsequent questions around these responses. Such questions may lead pupils step-by-step toward a desired conclusion or they may be open-ended to provide freedom and flexibility in the way learners approach problems. Encouraging children to talk about ideas gives teachers an inside view of thinking and also gives a child insight into his own thought processes. There is ego reinforcement when someone listens.

Acceptance of Varied Responses. In a class where discovery by pupils is valued, teachers must be willing to accept a variety of responses. For example, a teacher may ask: "Can you use a whole number to tell what part of the region is colored?" Some pupil may reply, "Yes, the number 3." Another may reply, "Yes, the number 5." Both pupils show understanding of the question.



The teacher must recognize this degree of understanding and proceed with other questions. The teacher may then ask: "Does the number 3 tell what part of the region is colored or does it tell the number of pieces?" "Does the number 5 tell the total number of pieces or the part that is shaded?" Through recognition of the thinking reflected in the variety of responses, the teacher leads pupils to see that the number $\frac{3}{5}$ shows the part that is shaded. The fractional number $\frac{3}{5}$ is identified by using the whole numbers 3 and 5 but is not a whole number itself.

Variety in Algorithms. In a class where understanding is valued, teachers encourage the use of longer algorithms. For example, the algorithms in A and in B are longer but may be more meaningful than the shorter one. When pupils are introduced to the compact algorithm, transition from the longer to the shorter form must be smooth and understandable for the learners. For some pupils transition to the more abbreviated algorithm C may be omitted completely without harm.

```
A         25
           5
          20
    42)1050
        840
        210
        210

    42)1050
        840   20
        210
        210    5
              25
```

```
B         25
    42)1050
        840
        210
        210
```

```
C         25
    42)1050
        84
        210
        210
```

Understanding Followed by Practice. Questions and classroom discussion are vital in developing understanding of an idea or procedure and are necessary predecessors of practice. Practice that follows should provide for varied use and application of the topic. Work done by a child must be checked carefully by the teacher as soon as possible to insure success for the learner and to provide the teacher with information for further planning. Systematic review work is necessary if the topic is to remain in the learner's repertoire.

Acceptance of Difference in Achievement. In any classroom teachers should expect wide variation in pupil responses and achievement. From some students the teacher should expect more insightful answers to questions and more astute observations. From others teachers should expect insight not so deep nor observations so profound. But from all pupils teachers can expect growth from year to year in mathematical maturity, competence in computation and ability in problem solving.

Development of Problem Solving Attitude. In mathematics it is a goal for pupils to develop a problem solving attitude. This attitude is encouraged and supported by a discovery oriented classroom where problems are posed by teachers and by pupils. If a problem engages pupils, the ensuing activity points in the direction of solving it. This is problem solving in its broadest sense.

Solution of Word Problems. A proper subset of the set of problems that engage pupils is the set of verbal or story problems which illustrate applications of mathematics. Applications are drawn from everyday experiences of pupils and of people they know. Where feasible, applications from other subject areas, such as science and social studies, are to be sought and exploited. With each problem the goal is to select the mathematical model that fits the practical situation. The mathematical model, such as addition, is often stated as a number sentence. *See Problem Solving*, page 153. The choice of a number sentence to express the mathematical model forces pupils to give attention to the analysis of the problem rather than plunging blindly into computation with the hope that the answer is correct. For many pupils, practical applications provide the prime motivation for studying mathematics. This phase of problem solving, then, must continue to receive emphasis in a sound elementary school program.

When enough time is spent on manipulation of concrete objects, discussion of ideas, utilization of varied pupil responses, solution of practical problems and appropriate practice, mathematical content makes sense. When content makes sense, the child becomes actively involved in the learning process and builds a firm foundation of competence on which to move ahead.

Evaluation, a Reflection of All Goals. Evaluation of the outcomes of the teaching of mathematics must give attention to all goals and must use techniques that focus on appropriate pupil behavior. If teachers aim to give attention to goals of understanding as well as to goals of computation, then evaluation must reveal degree of understanding as well as degree of skill in computation. If the development of active, or lifelong interest in the study of mathematics is a goal for elementary school children, then means for determining extent and growth of interest must be devised and used. If understanding and appreciation of the contributions of mathematics to civilization—past, present and future—is a goal, then avenues must be opened to enable children to communicate or demonstrate interest, understanding and appreciation.

Standardized tests and other paper and pencil tests should continue to be used to assess both understanding and skill in computation. Equally important is the use of observations and oral questions as children manipulate concrete materials, work independently on practice material or explore ways to solve problems. Interviews or discussions with individuals or small groups may reveal understanding of basic ideas or operational procedure, attitudes toward the study of mathematics, or appreciation of the part played by mathematics in the development of the modern world.

Evaluation of this type makes less demand on teachers to correct papers after school. Instead, the teacher assesses progress in the classroom by watching children at work, checking papers as work goes on and analyzing questions and oral responses. Evaluation which becomes an individual process related to the growth that a child has made and the help that is needed as he develops understanding and skill cannot be reduced to a score or letter grade that merely indicates the number of correct answers on a written lesson.

Evaluation, a Basis for Planning Instruction. The real purpose of evaluation is the assessment of strengths and weaknesses to enable the teacher to plan subsequent instruction more effectively. Machines and testing devices are helpful in collecting data on strengths and weaknesses, but no machine can make the kind of assessment which helps a child move ahead. Only a knowledgeable, sensitive teacher can do this.

MATHEMATICAL STRANDS WITH TEACHING SUGGESTIONS

SETS AND NUMBERS

A set is a collection of things so described or defined that it is possible to determine that a particular thing is a member or element of the set. Sets of objects have been used for many years to help children to count and to answer questions involving the idea of "how many." Not all sets have the same properties but all sets have the property of number. The idea of number is a central theme of the school mathematics program.

Ideas of sets and elementary notions of set theory are used to develop and expand the concept of number in this guide. All sets which can be matched in one-to-one correspondence share a common property of number. Numerals are names and symbols assigned to a particular number property.

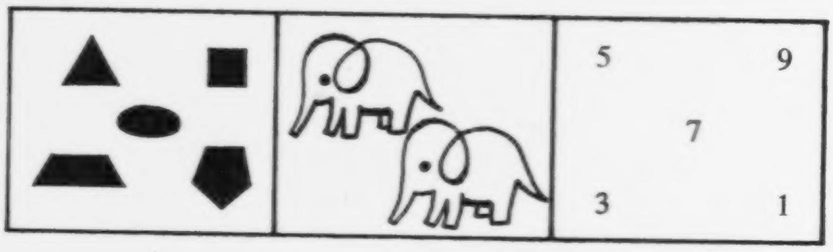
Numbers can be used in a cardinal sense to answer questions of how many. They can also be used in an ordinal sense to answer questions of which one. When elements of a set are arranged in a definite order, an ordinal number describes the position of an element in the given set, as a student is third in line or he is reading page 20, the twentieth page.

Although the study of number is extended in this guide to include the set of rational numbers, emphasis in elementary school is concentrated on the set of whole numbers and the set of fractional or positive rational numbers.

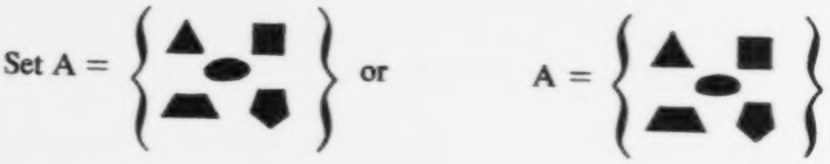
Mathematical Ideas | **Illustrations and Explanations**

The Meaning of a Set

A *set* is a well defined collection. The collection may consist of physical objects, alike or unlike, or abstract ideas, alike or unlike. The things in a set are usually called *elements* or *members*. A set is well defined if a given object or idea may be identified as belonging to or not belonging to the set.



Conventionally, set elements are enclosed in *braces* and the set is named by a capital letter.

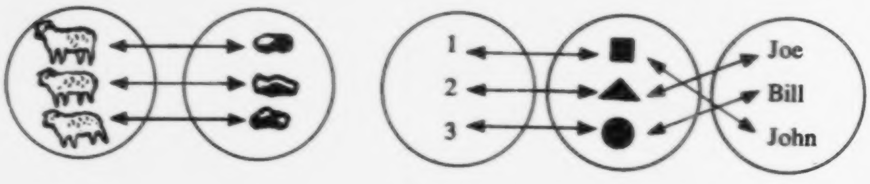


The elements of a set may be listed individually or identified by a descriptive phrase.

C = {5, 9, 7, 3, 1} C = {All odd numbers less than ten}

One-to-one Correspondence

Two sets A and B are in *one-to-one correspondence* if each element in A can be matched with a single element in B and each element in B can be matched with a single element in A.



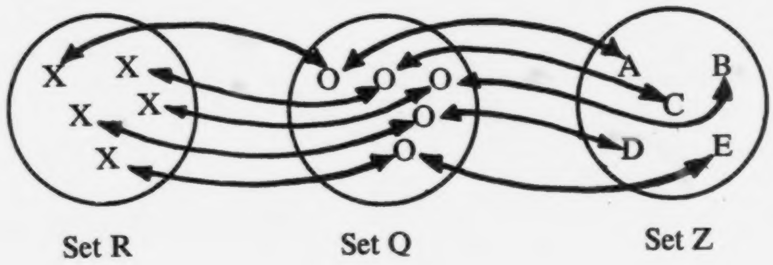
Set S Set P Set N Set C Set B

The members of Set S and Set P are matched in one-to-one correspondence.

The members of Set N, Set C, and Set B are in one-to-one correspondence with each other.

The fundamental basis for the concept of number is one-to-one correspondence or pairing of the elements of one set with the elements of another set. Before man devised a system of counting, he used one-to-one correspondence to keep records. Shepherds kept track of sheep by matching a pebble with a sheep. A record of time was kept by pairing each day with a mark on a stick or a stone.

One-to-one correspondence is the idea involved in set equivalence and can be used in developing a sound concept of number.



Number

The one property shared by all sets is the property of how many. This is called the *number property* of the sets.

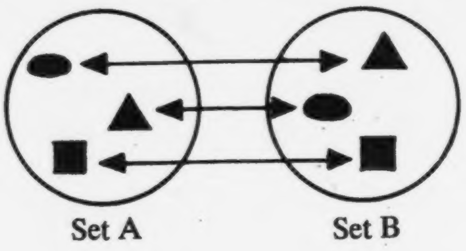
The number property of Set S above, for example, is 3.

The number property of Set R above is 5.

All other sets in one-to-one correspondence with Set S have the same number property or cardinal number, 3.

All other sets in one-to-one correspondence with Set R have the same number property or cardinal number, 5.

Equivalent sets: If two sets are in one-to-one correspondence, they are said to be equivalent. Equivalent sets have the same cardinal number.

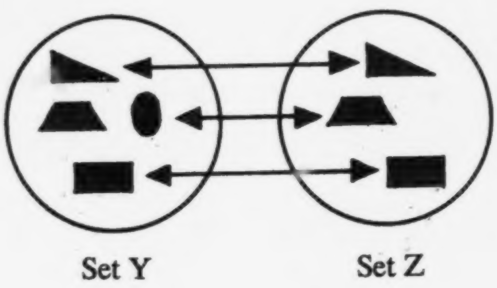


L {a, e, i, o, u}
↕ ↕ ↕ ↕ ↕
N {1, 2, 3, 4, 5}

Sets A and B are in one-to-one correspondence. The cardinal number of each is 3.

Sets L and N are in one-to-one correspondence. The cardinal number of each is 5.

Non-equivalent sets: If two sets are not in one-to-one correspondence, they are non-equivalent sets and have different cardinal numbers. The set containing more members has a cardinal number *greater than* the cardinal number of the other. The set with fewer members has a cardinal number *less than* the other.



Set Y has more members than Set Z. Set Z has fewer elements than Set Y. The cardinal number of Set Y is greater than the cardinal number of Set Z.

The comparison of the numbers may be shown by using the symbol $<$ to mean *is less than* and the symbol $>$ to mean *is greater than.*

$4 > 3$ Four is greater than three.

$3 < 4$ Three is less than four.

Also,

$$N(Y) > N(Z)$$
$$N(Z) < N(Y)$$

The N preceding the name of the set means "the number property of the set."

This use of N is not the same as the use of n as a variable in sentence such as $n + 5 = 7$.

When sets are arranged in order of *one more than,* the cardinal numbers of the sets are ordered and become the counting numbers.

{★} 1 {★, ★} 2 {★, ★, ★} 3 {★, ★, ★, ★} 4 {★, ★, ★, ★, ★} 5

Variables

A variable is a symbol that may be replaced by any element from a given set. Symbols such as □, △, ○ are used in early grades and letters such as *n, x,* or *y* in later grades.

Examples from the set of whole numbers:

$\square + 2 = 9$ This sentence is neither true nor false. If □ is replaced by 7, the sentence reads $7 + 2 = 9$ which is a true statement. Thus 7 is a solution to the *open sentence.*

The solution of a mathematical sentence with variables is the set of numbers that replaces the variables and makes the sentence true.

$y - 7 = 4$ If y is replaced by 9, the sentence reads $9 - 7 = 4$ which is false. Thus 9 is not an element of the solution set. However, if y is replaced by 11, the sentence reads $11 - 7 = 4$ and 11 is a solution.

When the variable in a sentence is replaced by an element from the given set, the sentence may be classified as "true" or "false."

$5 - 8 = n$ This sentence has no solution in the set of whole numbers. There is no whole number that can replace n such that the sentence will be true. Hence, the solution is the empty set.

Example from the set of integers:

$5 - 8 = n$ There is a number in the set of integers that can replace n such that the sentence is a true statement, namely $^{-}3$. Thus the solution is $^{-}3$ because $5 - 8 = {}^{-}3$.

When a given variable occurs more than once in a sentence, the same element is used for each occurrence of the variable.

Examples from the set of whole numbers:

$\square \times \square = 36$
$6 \times 6 = 36$
{ 6 }

$\square + \square = 12$
$6 + 6$
{ 6 }

When different variables occur, they may represent different numbers or the same numbers.

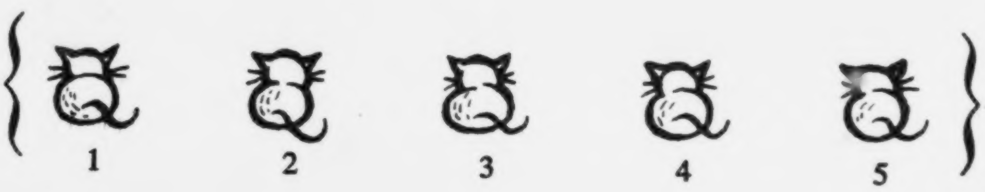
$\square + \triangle = 10$

Some pairs of numbers that make the sentence true are shown.

□	△
8	2
4	6
0	10
5	5

Counting

Counting is the process of finding how many elements there are in a set. In counting, the elements of a set are matched with the counting numbers beginning with one. The last name



The cardinal number of the set is "5."

said in one-to-one correspondence of number names with elements of a set is the *cardinal* number of that set. This cardinal number names the number property of the set.

As elements of a set are counted, each element is named in a definite order. The order or position of each element is designated by an *ordinal* number.

January is the *first* month of the year.

February is the *second* month of the year.

March is the *third* month of the year.

First, second, and third are ordinal numbers which specify the order in which months are named in the year.

Set Terminology

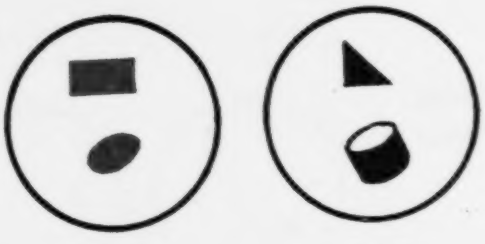
Equal sets: Two sets are equal if and only if they contain exactly the same members.

A = {★}	B = {★}	A = B
E = {1, 3, 5}	F = {5, 1, 3}	E = F
C = {2, 4, 6}	D = {3, 5, 7}	C ≠ D

Set C and Set D are not equal. However, Set C is equivalent to Set D since the elements of Set C can be matched in one-to-one correspondence with the elements of Set D.

> When the symbol "=" is used in mathematics, it means *is identical to* or *is exactly the same as*. For example, if A and B are sets, A = B means Set A is identical to Set B. For numbers, $5 + 3 = 8$ means $5 + 3$ represents the same number as 8 even though the numerals "5 + 3" and "8" are different. Likewise $\frac{1}{2} = \frac{2}{4}$ means that $\frac{1}{2}$ names exactly the same number as $\frac{2}{4}$ even though "$\frac{1}{2}$" and "$\frac{2}{4}$" are different fractions (symbols). The symbol ≠ is used to indicate *is not identical to* or *is not exactly the same as*.

Disjoint sets: Disjoint sets are sets which have no members in common.



Set A Set B

John
Jane
Alice

Set C

Bill
Nell
Jo

Set D

Empty set: The empty set is a set containing no members or elements. *Zero* is the number associated with the empty set and is its cardinal number.

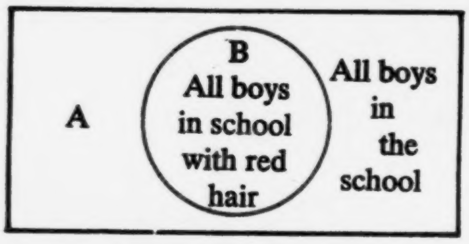
$\{\quad\}$

Set M

The Set M is an empty set.

Another symbol for the empty set is Ø.

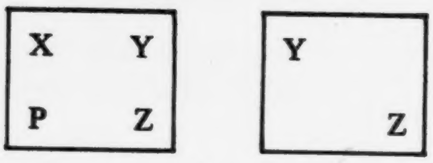
Subset: A subset of A is a set B such that each element of B is an element of A. Set B may contain every element of Set A. By agreement, the empty set is a subset of all sets.



A {Set of all boys in the school.}

B {Set of red-haired boys in the school.}

The set of red-haired boys is a subset of the set of all boys in school.



Set A Set B

Set B is a subset of Set A.

Finite set: A finite set is a set whose members can be counted with the counting coming to an end.

D = {All days in the week}

C = {Mary, Jane, Jim}

N = {All whole numbers less than 6}

Infinite set: An infinite set is a set whose elements cannot be counted so that the counting comes to an end. There is no last number.

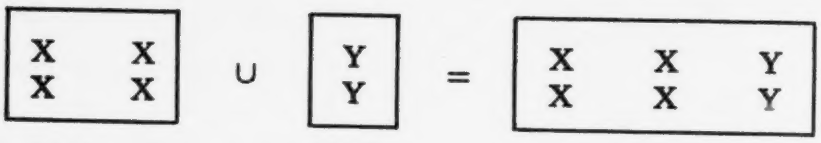
The set of all counting numbers

Set C {1, 2, 3, 4, . . .}

> The symbol . . . is placed after the last element of an infinite set to denote "continuation in this manner indefinitely."

Set Operation

Union of sets: The union of two sets is the set consisting of the elements contained in either set or in both sets.



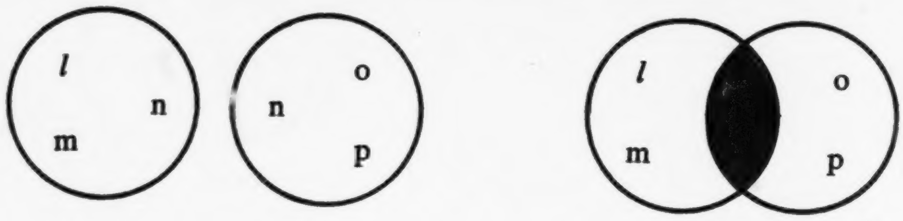
Set A Set B Set C

Sets are joined, not added, and the symbol for set union is ∪.

$\{xxxx\} \cup \{yy\} = \left\{\begin{matrix} xxx \\ xyy \end{matrix}\right\}$ Set A ∪ Set B = Set C

> The operation of joining disjoint sets is the foundation for addition.

If an element is a member of both sets, the element is listed only once in the union of two sets.



Set A Set B A ∪ B = {*l*, m, n, o, p}

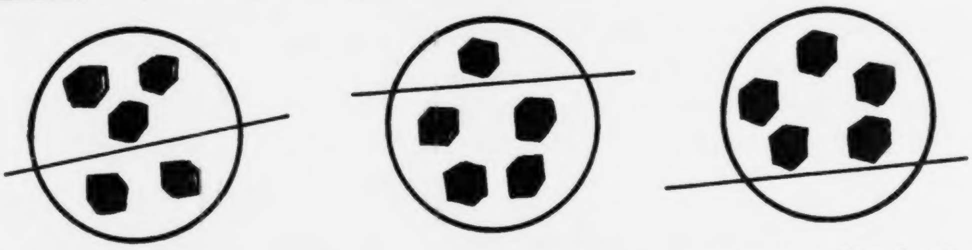
The union of a set with the empty set is the set itself.

{A, B, C} ∪ { } = {A, B, C}

The union of the empty set with another set may be used to establish zero as the identity element in addition.

Mathematical Ideas | **Illustrations and Explanations**

Separation of sets: Set separation involves partitioning members of a set to form subsets.

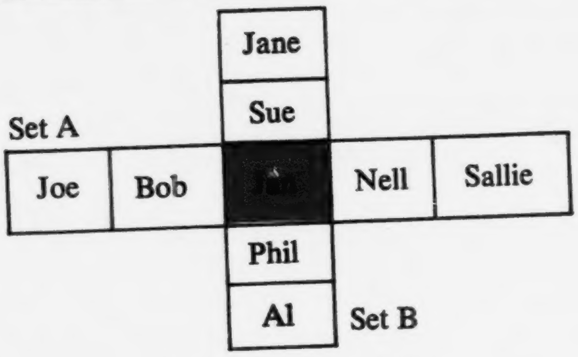


A set of five boxes may be separated into several pairs of subsets such as:

3 boxes and 2 boxes
4 boxes and 1 box
5 boxes and 0 boxes

Separating a set into two subsets may be used to introduce the idea of subtraction.

Intersection of sets: The intersection of two sets is a set consisting of the elements that are common to the two sets.

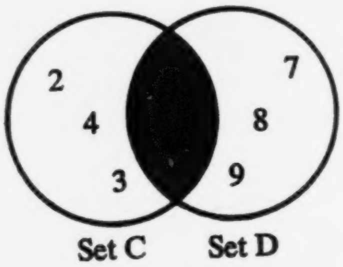


A = {Joe, Bob, Jan, Nell, Sallie}

B = {Jane, Sue, Jan, Phil, Al}

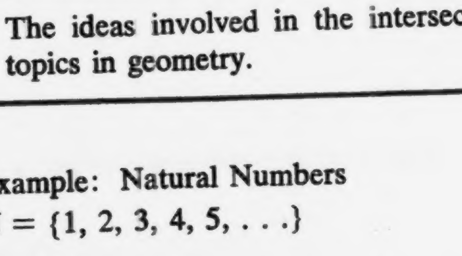
The symbol for intersection is ∩.

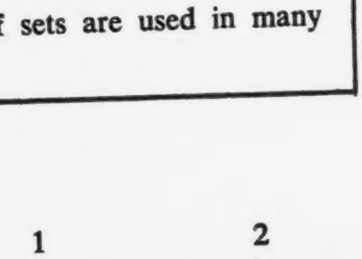
A ∩ B = Jan



C ∩ D = {5, 6}

The intersection of two disjoint sets is the empty set.





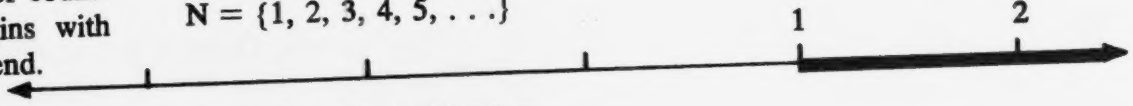
E ∩ F = { }

The ideas involved in the intersection of sets are used in many topics in geometry.

Summary of Sets of Numbers

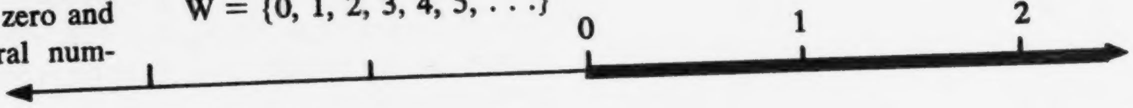
Counting numbers: The set of counting or *natural* numbers begins with one and continues without end.

Example: Natural Numbers
N = {1, 2, 3, 4, 5, . . .}



The set of whole numbers: The set of whole numbers consists of zero and all of the counting or natural numbers.

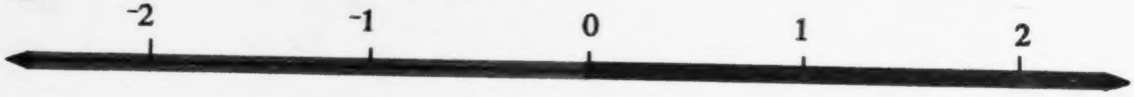
Example: Whole Numbers
W = {0, 1, 2, 3, 4, 5, . . .}



The set of integers: The set of integers consists of the set of whole numbers and the set of negatives of the natural numbers.

Example: Integers

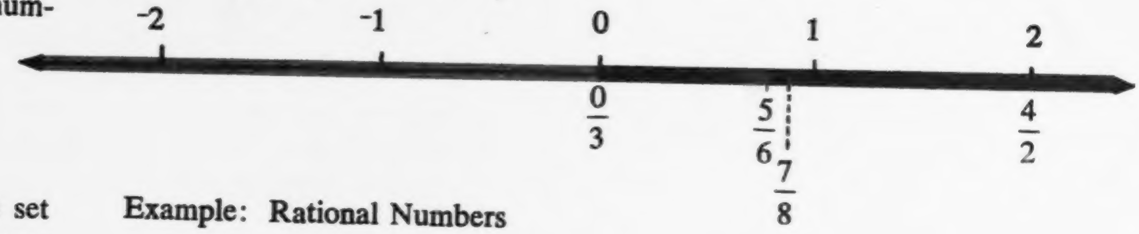
$I = \{\ldots {}^{-}2, {}^{-}1, 0, {}^{+}1, {}^{+}2, \ldots\}$



The set of fractional numbers: The set of fractional numbers consists of numbers which are written in the form $\frac{a}{b}$, where *a* and *b* are whole numbers and $b \neq 0$.

Example: Fractional Numbers

$$F = \left\{\frac{0}{3}, \frac{5}{6}, \frac{7}{8}, \frac{4}{2}, \ldots\right\}$$

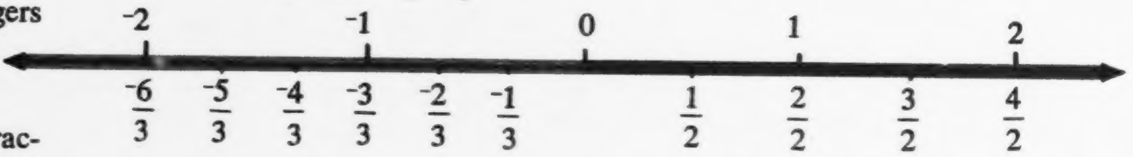


The set of rational numbers: The set of rational numbers consists of numbers which can be expressed in the form of $\frac{a}{b}$ where *a* and *b* are integers and $b \neq 0$.

The sets of counting numbers, the whole numbers, the fractional numbers and the negatives of all of these numbers are subsets of the set of rational numbers.

Example: Rational Numbers

$$R = \{3, {}^{-}1, \frac{3}{2}, \frac{1}{2}, 6, \frac{7}{6}, \frac{{}^{-}1}{3}, 0 \ldots\}$$



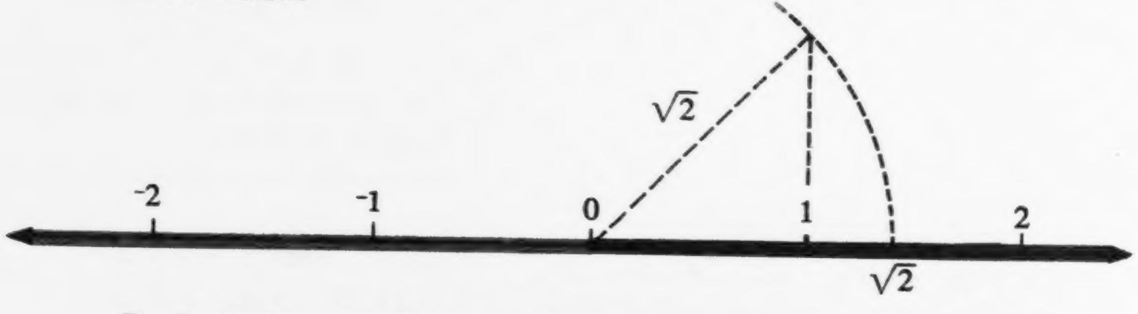
The set of irrational numbers: The set of irrational numbers consists of numbers that match points on a line and which cannot be expressed in the form $\frac{a}{b}$, where *a* and *b* are integers, $b \neq 0$.

Example: Irrational Numbers

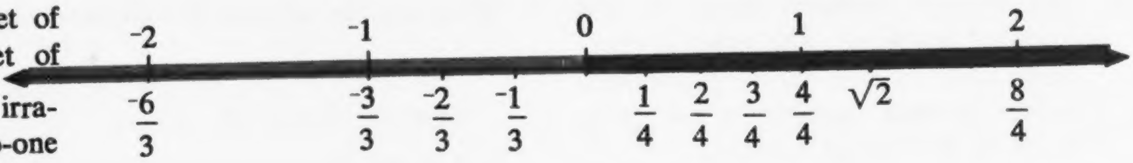
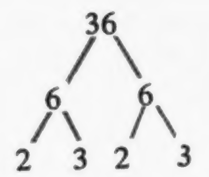
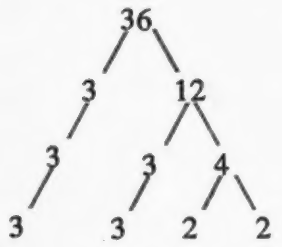
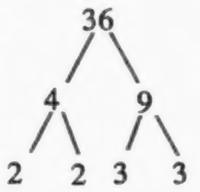
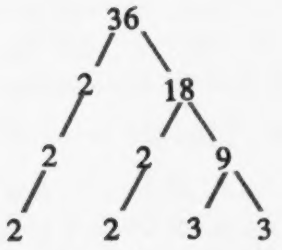
The symbol $\sqrt{\ }$ represents *the square root of.*

Since there is no rational number which when multiplied by itself, exactly equals 2, there is no rational representation of $\sqrt{2}$. However, a rational approximation can be made to any desired degree of precision.

There is a point on the number line that corresponds to every rational number. However, there are points on the number line that do not correspond to any rational number. One of these points is demonstrated below.



By the Pythagorean Theorem the sum of the squares of the sides of a right triangle is equal to the square of the hypotenuse. Applying theorem to the above triangle the hypotenuse is $\sqrt{2}$. Using a compass with radius equal to $\sqrt{2}$, and describing an arc to intersect the number line, the point of intersection is $\sqrt{2}$. This point of intersection corresponds to the number $\sqrt{2}$. This same point does not correspond to a rational number.

Mathematical Ideas	Illustrations and Explanations
The set of real numbers: The set of real numbers consists of the set of rational numbers and the set of irrational numbers. There is a one-to-one correspondence between real numbers and all points on the number line.	 -2 -1 0 1 2 -6/3 -3/3 -2/3 -1/3 1/4 2/4 3/4 4/4 √2 8/4 For each real number there is a point on the number line. For each point on the number line there is a real number.
Elementary Aspects of Number T	
Even numbers: A whole number that is the product of 2 and a whole number is called an even number.	Even numbers: {0, 2, 4, 6, 8, . . .} $0 = 2 \times 0$ $2 = 2 \times 1$ $4 = 2 \times 2$ $6 = 2 \times 3$ $8 = 2 \times 4$
Odd numbers: Any whole number that is not an even number is an odd number. If a whole number is divided by 2 and yields a remainder of 1, the number is an odd number.	Odd numbers: {1, 3, 5, 7, 9, . . .} $1 = (2 \times 0) + 1$ $3 = (2 \times 1) + 1$ $5 = (2 \times 2) + 1$ $7 = (2 \times 3) + 1$ $9 = (2 \times 4) + 1$
Prime numbers: A prime number is a counting number greater than 1 that has only itself and 1 as factors. With the exception of the number 2, all prime numbers are odd numbers. There is no greatest prime number.	The set of the first eight prime numbers = {2, 3, 5, 7, 11, 13, 17, 19.}
Composite numbers: A composite number is a counting number greater than 1 that is not a prime number.	The set of the first ten composite numbers = {4, 6, 8, 9, 10, 12, 14, 15, 16, 18.}
Prime factorization: Every composite number can be expressed as a product of primes in exactly one way except for the order in which the prime factors appear in the product. This is the fundamental theorem of arithmetic.	Pairs of factors of 36 are shown on the second line (from the top) of the factor tree. Each factor that is not prime is factored again. The prime factors of 36 are shown on the bottom row.     $36 = 2 \times 2 \times 3 \times 3 = 2^2 \times 3^2$

The greatest common factor of two counting numbers: The greatest common factor of two counting numbers is the greatest number which is a factor of *both* given numbers.

There are two ways to find greatest common factors (g.c.f.).

The g.c.f. of 12 and 18 is the greatest number which is a factor of both 12 and 18.

1. List the factors of both numbers and note the greatest factor common to them.

   Factors of 12: 1, 2, 3, 4, 6, 12

   Factors of 18: 1, 2, 3, 6, 9, 18

   Common factors: 1, 2, 3, 6

   g.c.f. is 6.

2. Factor each number into its prime factorization.

   $12 = 2 \times 2 \times 3$
   $18 = 2 \times 3 \times 3$

   The prime factors common to both numbers are 2 and 3. Then $2 \times 3 = 6$ is the g.c.f. of 12 and 18.

The least common multiple of a set of counting numbers: The least common multiple (l.c.m.) of a set of counting numbers is the least number that is a multiple of the numbers.

To find the least common multiple of two numbers:

1. List the counting number multiples of each number. Then note the least multiple common to the numbers.

   Example: Find the l.c.m. of 8 and 12.

   Counting number multiples of 8: 8, 16, 24, 32, 40, 48 . . .
   Counting number multiples of 12: 12, 24, 36, 48, 60 . . .

   The intersection of the two sets of multiples is the set of common multiples: 24, 48, 72, 96 . . . The l.c.m. is 24.

2. Factor each number into its prime factors. The l.c.m. is the product of the greatest common factor and the remaining factors.

   Example: Find the l.c.m. of 8 and 12.

   $8 = 2 \times 2 \times 2$
   $12 = 2 \times 2 \times 3$

   Greatest common factor is 2×2
   The remaining factors are 2 and 3
   l.c.m. = $(2 \times 2) \times 2 \times 3$
   l.c.m. is 24.

To find the l.c.m. of three numbers:

1. Find the l.c.m. of two of the numbers. Then find the l.c.m. of these two numbers and the third number. In a similar way, the l.c.m. of four or more numbers is found.

   Example: Find the l.c.m. of 8, 12, and 15.

   l.c.m. of 8 and 12 is 24
   l.c.m. of 24 and 15 is 120
   Hence, l.c.m. of 8, 12, and 15 is 120.

2. Divide the numbers of the set by successive prime numbers until all the quotients are one. If a number is not divisible by the prime number, repeat the number on the next line.

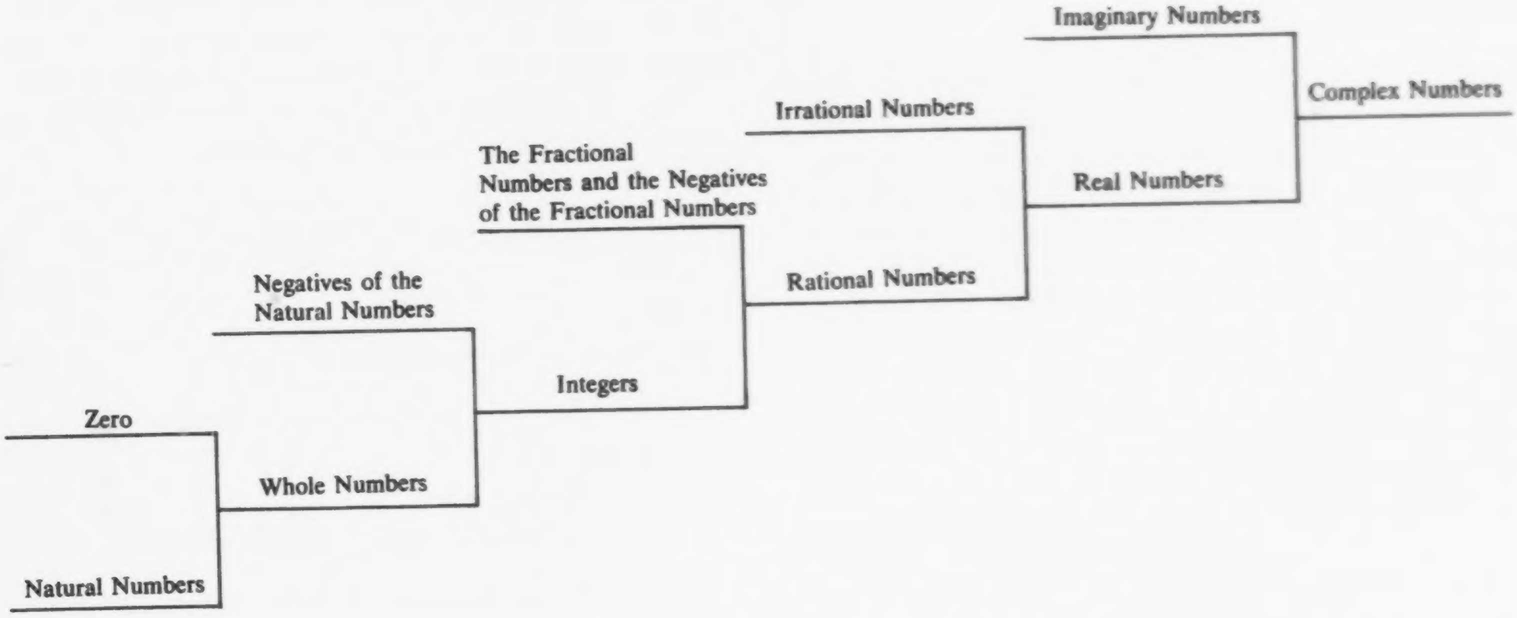
$$\begin{array}{r|ccc} 2 & 8 & 12 & 15 \\ \hline 2 & 4 & 6 & 15 \\ \hline 2 & 2 & 3 & 15 \\ \hline 3 & 1 & 3 & 15 \\ \hline 5 & 1 & 1 & 5 \\ \hline & 1 & 1 & 1 \end{array}$$

Then the l.c.m. is the product of the prime divisors.

$$\begin{aligned} \text{l.c.m.} &= 2 \times 2 \times 2 \times 3 \times 5 \\ &= 120 \end{aligned}$$

Most attention is given in the elementary school to the positive rational numbers and zero. Although the theory behind imaginary numbers is beyond the scope of a guide for elementary school mathematics, the set of imaginary numbers is included in the diagram below to indicate to teachers that later courses in mathematics may extend the study of numbers to encompass the entire system of complex numbers.

Diagram of the Complex Number System


Imaginary Numbers
Complex Numbers
Irrational Numbers
Real Numbers
The Fractional Numbers and the Negatives of the Fractional Numbers
Rational Numbers
Negatives of the Natural Numbers
Integers
Zero
Whole Numbers
Natural Numbers

TEACHING SUGGESTIONS FOR SETS AND NUMBERS

The Meaning of Set

Children enter school with some understanding of sets. From familiarity with a pair of shoes, a set of dishes, a bunch of carrots, a flock of birds, or a collection of rocks children recognize many synonyms for the word set. Sets are recognized as crayons are placed in boxes, books filed on shelves, pictures arranged on bulletin boards, dishes placed on trays or children grouped for games. Games may be developed to identify and describe many more sets in the environment. Familiar objects may be classified and grouped by size, shape, color or use.

Although a collection may consist of more than one object, the noun set is singular and refers to *one* group. It is use of the *idea* of set that is of real significance in elementary mathematics. Over emphasis or over formalization of either set vocabulary or symbols may, in fact, be detrimental in helping children to develop understanding of the important uses of sets in number work and in geometry.

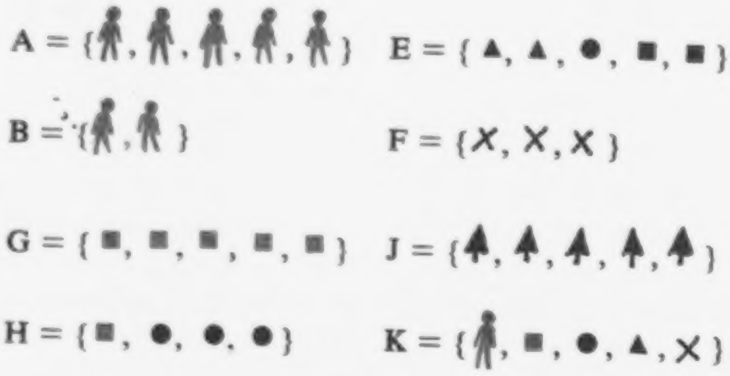
One-to-one Correspondence

As opportunity is provided for children to find their own chairs, distribute books or materials, match pairs of objects or objects to pictures, understanding of one-to-one correspondence develops. Matching fingers of a glove to fingers on a hand, of mittens to hands, of hands to eyes, of eyes to ears, of feet to shoes are examples of one-to-one correspondence with which a child is personally concerned. Many of the folk tales with which the child is familiar, such as "The Three Kittens," "Goldie Locks and the Three Bears," "Snow White and the Seven Dwarfs," capitalize on the idea of correspondence. As children dramatize these stories, they reinforce ideas of correspondence.

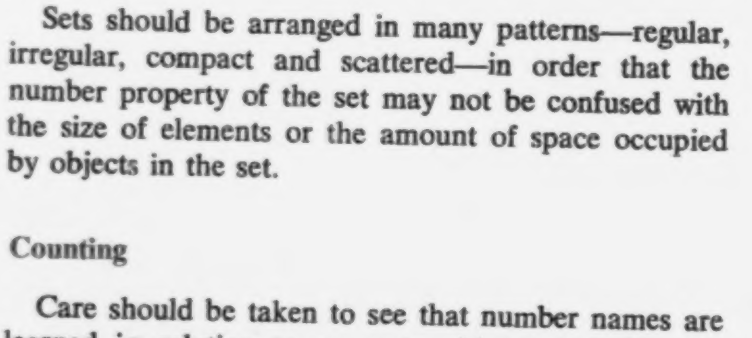
From manipulation of concrete objects teachers may move to semi-concrete representation on felt boards or bulletin board to extend understanding. Care should be taken to show that the one-to-one correspondence exists regardless of the position of elements within the sets. Elements of sets arranged compactly may be scattered or rearranged to present many different patterns. Density and arrangement tend to influence a child's perception of one-to-one correspondence. Variety in kinds and shapes of elements, arrangements of the elements and ways in which sets are matched broadens the concept of one-to-one correspondence. Equally important is the need to provide opportunities for observation and discussion as ideas of correspondence develop.

Number

Sound concepts of number develop slowly over long periods of time. Since concepts grow from individual manipulation and observation of physical objects, it is mandatory that provision be made for children to have many experiences with concrete and semi-concrete materials to develop understanding of the number property of sets.



Although ideas of equivalence and non-equivalence are developed almost simultaneously, it may be simpler to introduce non-equivalent sets first, particularly when the two sets are obviously non-equivalent. For example, begin with sets such as A and B where comparison is obvious and move to other examples as shown where careful matching is needed.



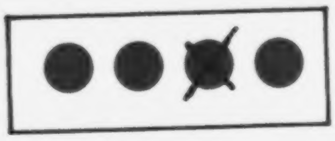
Sets should be arranged in many patterns—regular, irregular, compact and scattered—in order that the number property of the set may not be confused with the size of elements or the amount of space occupied by objects in the set.

Counting

Care should be taken to see that number names are learned in relation to concrete objects and that the number names are ordered. Many children enter school knowing number names and this knowledge should be recognized and utilized. Songs such as "Ten Little Indians," "One Potato," "This Old Man" and various counting rhymes and finger plays are helpful in ordering the names. Counting as children march utilizes rhythmic sense that reinforces learning of number names in order.

Although the terms cardinal and ordinal are not used in elementary school, it is important that children be able to use number in the cardinal sense of "how many" and the ordinal sense of "which one." In using number in the ordinal sense care must be taken to insure that children know where counting has started in establishing the order: left, right, up, down, front and back. The cafeteria line may be used to show the importance of a point of reference since it makes a difference when a child is fourth from the front or fourth from the back of the line.

The number name, four, may also be used in the ordinal sense of "which one." For example, the position of the marked object, counting from the left can be named as "three" or "third." The context in which names such as "three," "six," "ten," are used will indicate whether the number is used in an ordinal or a cardinal sense.



Set Terminology

Equivalent sets: See Number, page 7.

Equal sets: The idea of equality may be most effectively developed by rearranging elements in a given set to demonstrate that change in pattern in no sense changes the set since it still contains the same members. Each new arrangement forms a set that is equal to the original.

Empty sets: Meaning of the empty set may be developed through such activities as these:

1. Placing collections of materials in 4 of 5 containers (bags, covered boxes, jars, pockets); leaving one container with no members and pointing out that this set is called the empty set.
2. Asking children to describe the sets in the room which have no members. For example: the set of boys with green hair, the set of elephants, the set of girls two years old.

After the meaning of empty set is understood, zero is introduced as the cardinal number of the empty set. Activities similar to those described above may be used by asking children to give the number of the set, stressing the fact that the number of the set with no members is zero.

To develop further the idea that zero is the number of the empty set, there may be a discussion of a series of sets. Objects in a box may be counted to determine the number of elements in the set. Objects may be removed one at a time, counting after each removal to establish the number then in the set. When all elements have been removed, zero is established as the number name of the set with no members, or the empty set. "Zero," not "none," is the answer to the question, "How many members in the empty set?"

Subsets: Such activities as the following may be used to develop the idea of subsets:

1. Using children in the room as members of a set then asking the subset of children wearing glasses to raise hands; the subset of children with blond hair to stand; the subset of children with orange eyes to come to the front of the room, etc.
2. Separating sets of objects into a variety of subsets.
3. Using yarn to encircle various subsets of felt objects on the flannel board.
4. Describing the members of subsets of a given set. For example, naming members of the subset of months from June to December.

Infinite sets: While it is not important to teach the terms finite and infinite, these ideas are important in developing the concept that there is no greatest counting number. Even primary children can understand that there is always a next greater number.

In upper elementary grades children may be asked to name the greatest number they know, then to determine whether there is a greater number. After discussion of this they discover that counting goes on and on.

Set Operation

Union of sets: The union of disjoint sets may be illustrated by:

1. Having four boys join three girls to form a new set of children.
2. Moving two rows of desks together to demonstrate joining of sets.
3. Stringing sets of beads.
4. Joining sets of objects or pictures on the flannel board or on the magnetic board.

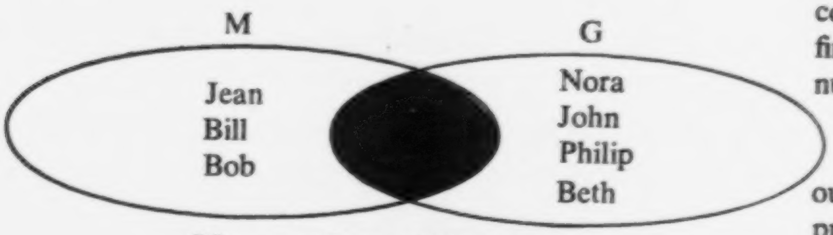
Later children form the union of sets which are not disjoint and may use sets whose members are words, numbers, or geometric figures. For example:

A = {1, 3, 5, 7, 9}   B = {3, 6, 9}

A ∪ B = {1, 3, 5, 6, 7, 9}

Intersection of sets: The intersection of sets may be illustrated by:

1. Asking children in row one to raise hands, asking children in column one to stand, and discussing the one who is both standing and raising his hand to demonstrate that this child is in the intersection.
2. Discussing activities in which pupils participate outside of the classroom: band, glee club, science club, math club, or library club and listing children who belong to the various groups. Diagrams may be made to illustrate set intersection. For example:



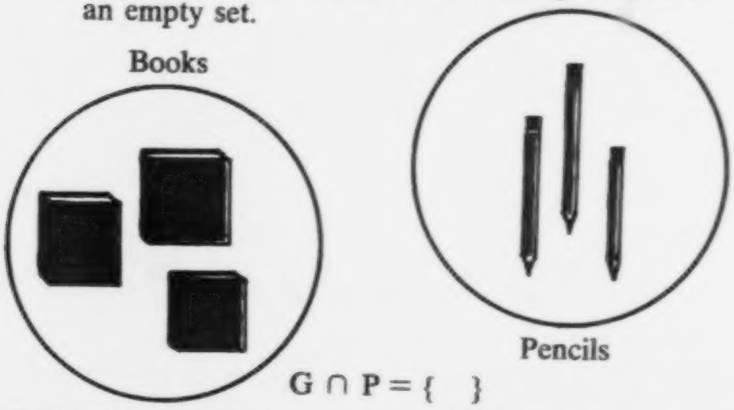
M = members of Math Club

G = members of Glee Club

A ∩ B = {Mary, Joe}

Mary and Joe belong to the intersection of sets M and G.

3. Using a set of books with a set of pencils to show that the intersection of two disjoint sets is an empty set.



G ∩ P = { }

Set union and separation are introduced in early grades as foundation for the operations of addition and subtraction. Excessive time spent on set symbolization is to be avoided.

Separation of sets: Ideas relative to the separation of sets may be developed by:

1. Giving each child a set of counters; asking that these counters be separated into two sets.
2. Placing objects on the flannel board, and using a ruler, pencil, or string to show separation into subsets.

Summary of Sets of Numbers

This material is included to help teachers understand that there is relationship between sets of numbers studied in elementary school and sets of numbers which may be studied later. Intentionally, no teaching suggestions are included in this section. Suggestions relative to teaching those sets of numbers pertinent to elementary mathematics will be found in other sections of the guide.

Elementary Aspects of Number Theory

Prime and Composite numbers: An interesting method of separating the prime numbers from the composite numbers is the Sieve of Eratosthenes. To find prime numbers less than 50, first list all counting numbers greater than 1 and less than 50:

Draw a ring around each prime number and cross out each composite number. In this sense only the prime numbers will not pass through the sieve. To begin, draw a ring around 2 and cross out each multiple of 2 thereafter. (If a number greater than 2 is a multiple of 2, it cannot be prime since it has 2 as a factor.)

	②	3	~~4~~	5	~~6~~	7	~~8~~	9	~~10~~
11	~~12~~	13	~~14~~	15	~~16~~	17	~~18~~	19	~~20~~
21	~~22~~	23	~~24~~	25	~~26~~	27	~~28~~	29	~~30~~
31	~~32~~	33	~~34~~	35	~~36~~	37	~~38~~	39	~~40~~
41	~~42~~	43	~~44~~	45	~~46~~	47	~~48~~	49	

Next, encircle 3 as a prime number and cross out all multiples of 3 that remain.

②	③	5	7	~~9~~
11	13	~~15~~	17	19
~~21~~	23	25	~~27~~	29
31	~~33~~	35	37	~~39~~
41	43	~~45~~	47	49

Next, encircle 5 and cross out all multiples of 5 that remain.

	②	③	⑤	7	
11		13		17	19
		23	~~25~~		29
31			~~35~~	37	
41		43		47	49

Of the remaining numbers, encircle 7 and cross out its multiples.

	②	③	⑤	⑦	
11		13		17	19
		23			29
31				37	
41		43		47	~~49~~

All the remaining numbers are prime numbers and may be encircled. This can be verified by attempting to factor them.

In similar fashion, the set of prime numbers less than any specified number can be determined.

NUMERATION

A numeration system is a means of naming numbers. It involves a set of symbols or *numerals* and rules for combining these numerals to name numbers. An efficient system of numeration is a significant achievement for any people. It evolves as the result of ages of experimental manipulation of concrete objects, tally marks, and symbols of various kinds. The ability to work with abstract symbols as representations of real objects is a distinguishing characteristic of civilized man.

A section on numeration is included in this guide primarily to call attention to five features of the Hindu-Arabic decimal system which make it an efficient system of numeration:

1. The Hindu-Arabic decimal system uses a base of ten.
2. The system employs a finite set of symbols represented by the numerals or digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.
3. A place value based on powers of ten, increasing in value from right to left, is assigned to each position in the numeral to represent numbers larger than 9.
4. Each digit in the numeral represents the product of the number it names and the place value assigned to its position.
5. Each number named is the sum of the products mentioned above.

While other systems may have some of these characteristics, it is important to remember that the Hindu-Arabic system includes all five. It is also important to note that a place value system makes it possible to extend the numeration system to the right of the ones place so that any rational number may be represented.

The study of other systems of numeration is included in this guide as a means of providing deeper understanding and appreciation of the efficiency of the Hindu-Arabic system. These include systems that use a base other than ten, those that use different sets of symbols, and those which do not employ place value. Use of bases other than ten is to be found in tables of measurements in daily use: Base sixty in sixty seconds make one minute, sixty minutes make one hour; base twelve in twelve inches make one foot; base sixteen in sixteen ounces make one pound. Systems that use different sets of symbols are represented by Roman numeration and systems which do not employ place value by the Egyptian system.

Mathematical Ideas	Illustrations and Explanations
Hindu-Arabic Decimal Numeration System	
Symbols: Basic symbols for numeration are distinct, single characters assigned to each whole number zero through 9.	The finite set of symbols for the Hindu-Arabic system is {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}. These unit symbols are commonly called *digits.*
Base: The Hindu-Arabic system employs the use of a combination of two or more symbols when recording numbers above 9. The numeral for the number which is one greater than 9 is called ten (10) and becomes the base of the system. The Hindu-Arabic system is called a *decimal system* of numeration because it uses a base of ten.	Example: 1, 2, 3, 4, 5, 6, 7, 8, 9, base. Using the first digit and 0, the base becomes 10 which means 1 base and no units. In other numeration, the system might be: 1, 2, 3, 4, base, or 1, 2, 3, 4, 10 (1 base and no units).

Place Value: The Hindu-Arabic system of numeration in addition to the idea of base also employs a positional pattern in naming any number greater than 9. The numeral for ten and all succeeding numerals of two or more digits are combinations made according to this pattern. In the numeral 10 the digit 1 occupies the position of 1 base and zero occupies the units' position.

The numerals from ten to nineteen have been given unique names.

Example:

10	1 base or (1 × 10) + 0	Ten
11	(1 × 10) + 1	Eleven
12	(1 × 10) + 2	Twelve
13	(1 × 10) + 3	Thirteen
14	(1 × 10) + 4	Fourteen
15	(1 × 10) + 5	Fifteen
16	(1 × 10) + 6	Sixteen
17	(1 × 10) + 7	Seventeen
18	(1 × 10) + 8	Eighteen
19	(1 × 10) + 9	Nineteen

Unique names derived from names for 2-9 have been assigned to each multiple of ten (the base) from ten to ninety.

20	2 bases or (2 × 10) + 0	Twenty
21*	(2 × 10) + 1	Twenty-one
30	(3 × 10) + 0	Thirty
40	(4 × 10) + 0	Forty
50	(5 × 10) + 0	Fifty
60	(6 × 10) + 0	Sixty
70	(7 × 10) + 0	Seventy
80	(8 × 10) + 0	Eighty
90	(9 × 10) + 0	Ninety
100	(10 × 10)	Hundred

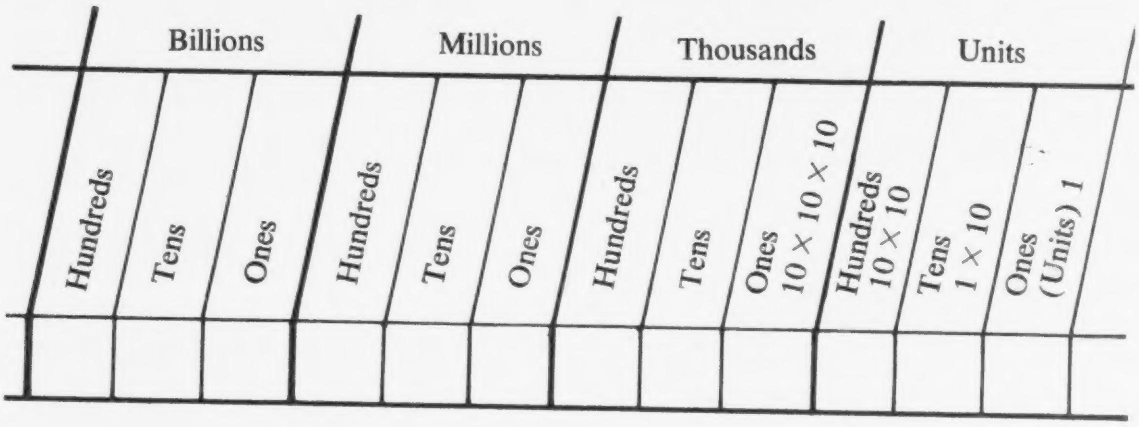
*The pattern used in counting from 1-9 continues with each multiple of 10 beyond 20.

Mathematical Ideas | **Illustrations and Explanations**

Unique names have been assigned to each group of ten tens or hundreds as each succeeding position to the left becomes 10 times the place value to its right.

Example:

1,000 (10 × 10 × 10)	Thousand
10,000 (10 × 10 × 10 × 10)	Ten thousand
100,000 (10 × 10 × 10 × 10 × 10)	Hundred thousand



Two ideas are involved in the place value principle:

1. There is a number assigned to each position in the numeral.
2. Each digit represents the product of the number it names and the place value assigned to its position.

Zero: Zero is the numeral used to denote the idea of "not any."

Zero is the number of the empty set.

$A = \{\ \};\ N(A) = 0$

Zero matches the point on the number line which separates the set of positive real numbers from the set of negative real numbers.

Zero is necessary in a numeration system which uses place value. Where 0 appears in a numeral, it indicates not any groups in the place which is held by the zero.

Example: In the numeral 403, the zero indicates that there are no groups of ten in the numeral. The numeral, therefore, names four groups of one hundred, no groups of ten, and three ones or units.

Exponents: In the base ten decimal system of numeration factors of ten play an important role. 10 × 10 can be written 10^2, where the 2, called the exponent, indicates how many times 10, the base, is used as a factor.

For example: $10^2 = 10 \times 10$

$10^3 = 10 \times 10 \times 10$

$10^4 = 10 \times 10 \times 10 \times 10$

$10^5 = 10 \times 10 \times 10 \times 10 \times 10$

For any number *a*, a^n indicates that *a* has been used as a factor *n* times.

10,000	1,000	100	. . .
10 × 10 × 10 × 10	10 × 10 × 10	10 × 10	. . .
10^4	10^3	10^2	. . .

Following this pattern, 10^1 is 10 and 10^0 is 1.

Any number to the first *power* (having an exponent of 1) is equal to the number.

Example: $10^1 = 10 \quad 15^1 = 15$
$5^1 = 5 \quad 8^1 = 8$

For any number a, $a^1 = a$

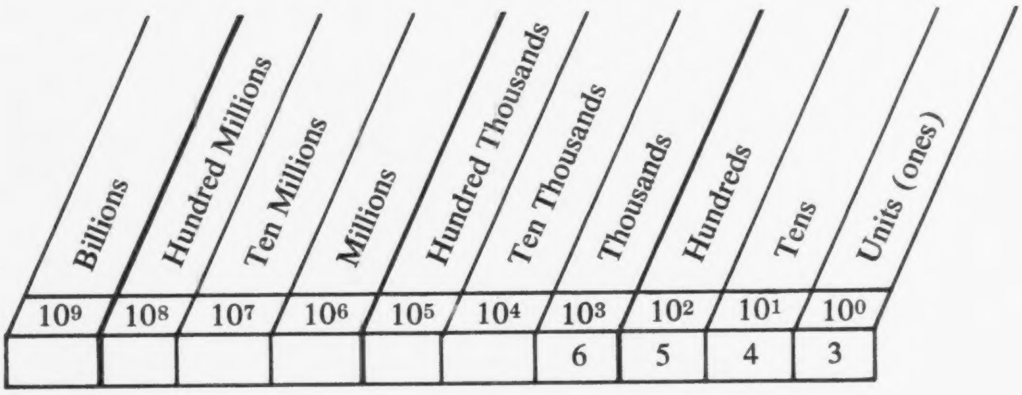
Any number (except zero) to the zero power (having an exponent of zero) is equal to one.

Example: $10^0 = 1 \quad 15^0 = 1$
$5^0 = 1 \quad 8^0 = 1$

For any number a, $a \neq 0$, $a^0 = 1$

In summary:

$$\begin{aligned} 6{,}543 &= 6{,}000 + 500 + 40 + 3 \\ &= (6 \times 1{,}000) + (5 \times 100) + (4 \times 10) + (3 \times 1) \\ &= 6 \times (10 \times 10 \times 10) + 5 \times (10 \times 10) + (4 \times 10) + 3 \times 1 \\ &= (6 \times 10^3) + (5 \times 10^2) + (4 \times 10^1) + (3 \times 10^0) \end{aligned}$$


Billions
Hundred Millions
Ten Millions
Millions
Hundred Thousands
Ten Thousands
Thousands
Hundreds
Tens
Units (ones)
10^9
10^8
10^7
10^6
10^5
10^4
10^3
10^2
10^1
10^0
6
5
4
3

Non-decimal Bases

The important features of place value systems of numeration are:

1. A base
2. The use of a finite set of symbols
3. The use of place value
4. The use of zero

Example: The numeral 101_{two} is read, "one zero one, base two," and indicates that instead of grouping by ones (units), tens, and ten tens, groups of ones, twos and two twos are used.

Example: $101_{two} = 1 \times (2 \times 2) + (0 \times 2) + (1 \times 1)$

Note: Zero and one are the only symbols employed in base two.

The Hindu-Arabic system of numeration uses the number ten as a base. The choice of ten as a base is arbitrary. Any other number, except zero and one, can be used as a base.

Example: $231_{four} = 2 \times (4 \times 4) + (3 \times 4) + (1 \times 1)$

Note: 0, 1, 2, 3 are symbols used in base four.

Examples: $987_{twelve} = 9 \times (12 \times 12) + (8 \times 12) + (7 \times 1)$
$5e6_{twelve} = 5 \times (12 \times 12) + (e \times 12) + (6 \times 1)$

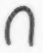
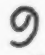
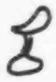
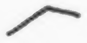
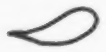
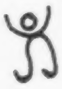
Note: 0, 1, 2, 3, . . . 9, t, e are symbols used in base twelve.

Work in non-decimal bases should be limited. Its value lies in helping students better understand the decimal system of notation by emphasizing the ideas of place value irrespective of base.

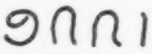
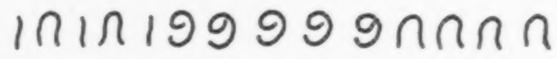
Ancient Numeration Systems

The Egyptian numeration system: The Egyptian numeration system used the number ten as a basis for grouping (base ten). Place value was not employed since the numeration system had no symbol for zero. Each group of ten and each group of ten tens, etc. required the creation of different symbols.

Example:

Hindu-Arabic	Egyptian	Names For Egyptian Symbols
1		stroke
10		heel bone
100		coil of rope
1,000		lotus flower
10,000		bent reed
100,000		burbot fish
1,000,000		astonished man

Numerals:

Hindu-Arabic	Egyptian
121	
563	

Order makes no difference since place value is not involved.

The Roman numeration system: The Roman numeration system used a base ten but added symbols for five, fifty, and five hundred so that numbers could be recorded with fewer repetitive symbols. Otherwise the Roman system was analogous to the Egyptian system.

Hindu-Arabic	Roman
1	I
5	V
10	X
50	L
100	C
500	D
1,000	M
10,000	$\overline{X}$
34,029	$\overline{XXXIV}$XXIX

Mathematical Ideas

In order to simplify further the recording of numbers the Roman system did two things:

Employed the subtraction principle by substituting IV for IIII and XL for XXXX.

Placed a bar over a part or all of a Roman numeral to indicate multiplication by 1,000.

Illustrations and Explanations

Numerals:

Hindu-Arabic		Roman
463	=	CDLXIII
400	=	CD
60	=	LX
3	=	III

The Roman system has a practical significance since it is used for recording dates, in numerals on clocks, numbering chapters in books, and in outlining.

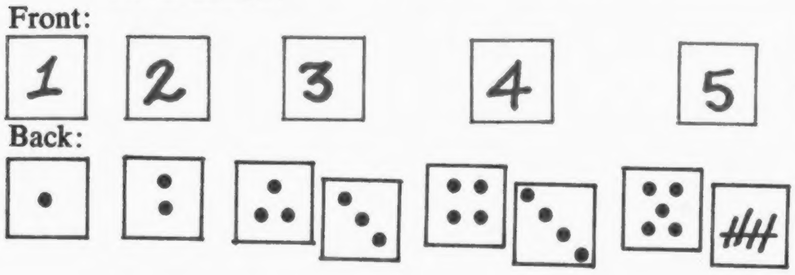
TEACHING SUGGESTIONS FOR NUMERATION

Hindu-Arabic Numeration System

Symbols: Many children are able to count sets of at least five objects when they enter school and some may even recognize a few number symbols. The extent of this knowledge is easily determined. The problem then becomes one of associating number symbols, both words and numerals, with the number property of a set.

Ability to associate symbol with number may be extended through such activities as the following:

1. Beginning with sets with a known number of objects and matching them with cards showing the numerals:



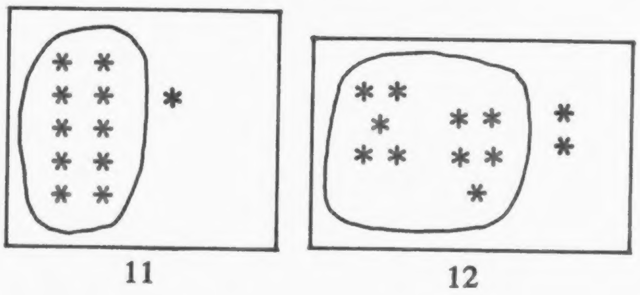
(Later word names may be added to the backs.

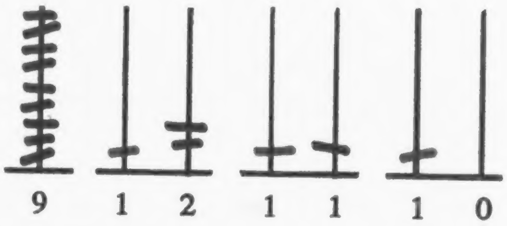
A set of cards should be available for each child involved in the activity.)

2. Providing stencils for children to cut out sets of numerals to use in games and activities like those which follow.
3. Playing games where teacher holds up a set of fingers and children match with the correct numeral; where teacher holds up a numeral and children match with sets of fingers.
4. Giving a numeral card to each child to find a place in line.
5. Giving each child a group of numeral cards to arrange in order from least to greatest, or greatest to least, or top to bottom.
6. Giving older children numeral cards of fractional numbers with like, then unlike denominators, or decimal fractions to arrange in sequential order.
7. Using old calendar pages and drawing a ring around:
   - a. Different numerals as dictated
   - b. A numeral between two numerals: 3-5, 8-10, 15-17
   - c. The numeral that represents the even number between 9 and 6, or 24 and 28
   - d. The numeral between 16 and 26 which represents a number divisible by 7, or which has 7 as a factor.
8. Giving time for children to invent games.
9. Writing numerals:
   - a. In order from 1 through 9
   - b. To make price tags
   - c. To play games
   - d. To keep score.

Base: Having learned to associate one of the numerals 0-9 with the number property of the set which it matches, children are ready to learn how numbers larger than 9 are represented. The word ten has been associated with the number of fingers on normal hands or the number of toes on normal feet, but numerals which say one more than nine or two more than nine may not be recognized.

Children work with such concrete materials as sticks, crayons, blocks, acorns, paper clips, or other objects which they have collected. It may prove helpful to begin with objects that may be grouped as 1 ten and 1 one, then as 1 ten and 2 ones, before introducing 1 ten and 0 ones. The next step, the use of bead frames, counting boards and abacuses, should be followed by use of semi-concrete materials, pictures or cut outs on flannel board, to illustrate grouping by ten or base ten.



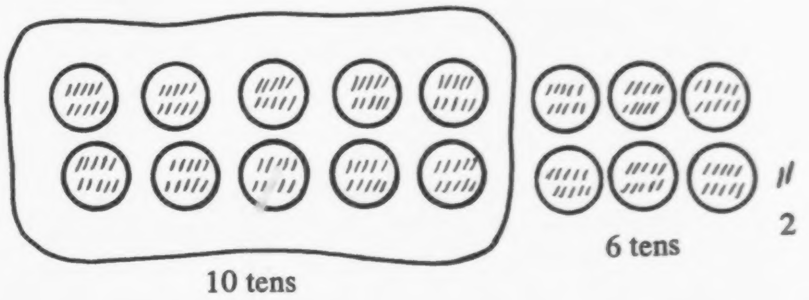


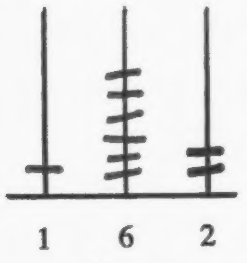
It is agreed that one mark or bead on left hand rod represents 10 marks or beads on the right hand rod.

Place value: Understanding of place value may be developed by establishing a place to the left for bundles of ten sticks each and to the right for any remaining sticks. One bundle of ten sticks and one single stick may be recorded as 1 ten and 1 one. By adding bundles to the tens side and single sticks to the ones side, larger numbers may be represented. These numbers may then be represented on the abacus or bead frame and notation recorded. It must be emphasized that numerals are read from left to right across the page.

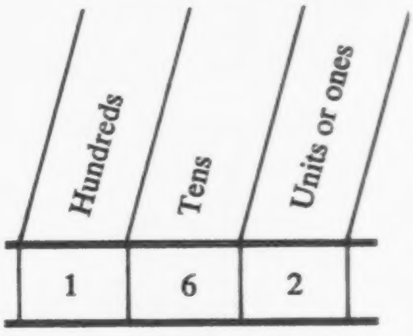
Making place value charts can re-enforce the understanding of place value. It is most important that place value demonstrations on a bead frame be seen by the class in the proper left to right perspective.

When ten bundles of ten accumulate, these are bound together to make one large bundle of ten tens. Procedure then is as with tens and ones; recording, demonstrating on the abacus and transferring notation to the number chart, 1 hundred, 6 tens, 2 ones, etc.





Abacus



Number Chart

When the idea of place value is established, numbers may be written in expanded notation as:

6 tens, 2 ones = 62 = 60 + 2

3 hundreds, 6 tens, 2 ones = 362 = 300 + 60 + 2

1 thousand, 3 hundreds, 6 tens, 2 ones = 1,362 = 1000 + 300 + 60 + 2

Children may work independently to establish understanding of place value using such activities as the following:

1. Indicating the value of each digit in a numeral: For example: In the numeral 365,198:

The 3 names three hundred thousand	or	300,000
The 6 names sixty thousand	or	60,000
The 5 names five thousand	or	5,000
The 1 names one hundred	or	100
The 9 names nine tens (ninety)	or	90
The 8 names eight ones (eight)	or	8

2. Using any five digits, such as 8, 3, 0, 9, 7, to represent the largest possible number and the least possible number not beginning with 0.

3. Playing such games as the following to re-enforce and evaluate understanding of place value:

   a. Circle the 5 which represents the largest number in each of the numerals:

      5345

      2545

      5555

   b. Order the numbers from largest to smallest.

4. Writing numerals using any three digits, such as 1, 2, 3 repeating use of each in as many places as desired. For example:

123	1,123	13,231
321	3,321	232,111
231	2,312	111,222 . . .

This activity can lead to the realization that an infinite set of numbers can be represented by using a finite set of symbols.

Zero: Activities described for place value illustrate a use of zero in a place value numeration system. When recording numerals which represent the number of objects grouped in ones, tens, hundreds, etc., or which represent a number illustrated on the abacus, attention should be called to the use of 0. For example, just as 7 is used to indicate the number of hundreds, zero is used to indicate that there are zero tens in seven hundred three, 703, and to indicate zero hundreds in one thousand eighty-nine, 1,089. Understanding zero as the cardinal number of the empty set should be recalled and utilized.

Exponents: Exponents provide a convenient way to record the powers of the base and the idea may be developed by building on understanding of place value notation.

100 is 10 tens or 10 × 10 which may be written 10^2

1,000 is 10 × 10 × 10 or 10^3

10,000 is 10 × 10 × 10 × 10 or 10^4

$10^5 = 100{,}000$
$10^4 = 10{,}000$
$10^3 = 1{,}000$
$10^2 = 100$
What next?
$10^1 = 10$
$10^0 = 1$

Opportunity may be provided for children to write numerals in expanded notation using exponents:

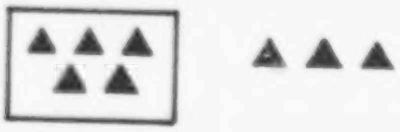
$$429 = 400 + 20 + 9 = (4 \times 100) + (2 \times 10) + (9 \times 1) = (4 \times 10^2) + (2 \times 10^1) + (9 \times 10^0)$$

$$6{,}097 = 6{,}000 + 000 + 90 + 7 = (6 \times 1{,}000) + (0 \times 100) + (9 \times 10) + (7 \times 1) = (6 \times 10^3) + (0 \times 10^2) + (9 \times 10^1) + (7 \times 10^0)$$

Non-Decimal Bases

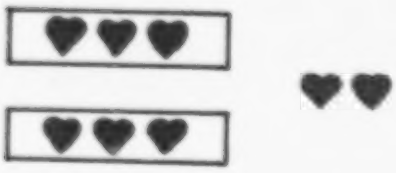
Non-decimal numeration systems are introduced to re-enforce the understanding of the concepts of base and place value.

Children may group objects and discuss the groupings. For example:



1 five and 3 ones

1 3 = 13_{five}



2 threes and 2 ones

2 2 = 22_{three}

Groupings may then be recorded in a place value chart. For example:

Twenty-fives	Fives (Base)	Ones
	1	3

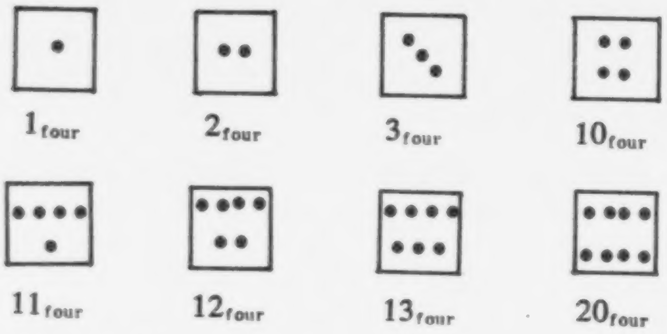
=

Nines	Threes (Base)	Ones
	2	2

=

Hundreds	Tens (Base)	Ones
		8

Symbols for other bases may be introduced. For example: Using base four and a single object, write 1_{four}. By adding one object at a time, children see how the base four symbol names each number of objects. When four objects are bundled, the use of place value in the notation for "one four and 0 ones (10_{four}) should be discussed. Continue adding objects and recording the symbol for each number of objects:



Bundle objects when necessary. This process may be continued until four fours are bundled into a sixteen. The numerals may be recorded in a place value chart similar to those above.

Place value charts may be built in other bases. Children may use what they know about place value to suggest how to write the numerals. As numerals are recorded, the part played by base and place value should be emphasized.

Enrichment activities for some pupils could include:

1. Making a calendar using different bases for each month.
2. Writing age, weight, height, and birth date in other bases.
3. Completing the blanks in such exercises as the following:

44_{five} = (4) fives (4) ones

32_{six} = (3) sixes (2) ones

201_{three} = (2) nines (0) threes (1) ones

$\underline{(42_{seven})}$ = 4 sevens 2 ones

$\underline{(236_{eight})}$ = 2 sixty-fours 3 eights 6 ones

$\underline{(23_{sixteen})}$ = 2 sixteens 3 ones

4. Writing base ten numerals for:

$23_{five} = \underline{(13)}_{ten}$

$111_{three} = \underline{(13)}_{ten}$

$36_{eight} = \underline{(30)}_{ten}$

$104_{six} = \underline{(40)}_{ten}$

$101_{two} = \underline{(5)}_{ten}$

$321_{four} = \underline{(57)}_{ten}$

The Egyptian Numeration System

By comparing the Egyptian numeration system with a place value system pupils can see advantages of a place value system of numeration. Symbols employed by the Egyptians may be introduced with emphasis on the base ten idea in the system and on the fact that symbols were repeated because the system did not use place value.

Children with special interest in ancient numeration systems may:

1. Write base ten numerals for given Egyptian numerals.
2. Learn more about the Egyptian numeration system from reference books.
3. Look in museums for samples of Egyptian writings containing numerals.

The Roman Numeration System

The Roman numeration system may also help children to appreciate the efficiency of a place value decimal numeration system. After the introduction of the different symbols a discussion of the subtraction principle involved is necessary. Roman numerals can then be compared with Egyptian as well as with the Hindu-Arabic symbols.

Discussion of ways Roman numerals are used today will include dates on buildings or movies; numerals on clocks; designations for outline or chapter headings or volumes of books.

Practice in writing Roman numerals and in converting Roman numerals to other systems may be used as enrichment activities or as teachers see practical need for this experience.

Pupils interested in demonstrating the efficiency of the Hindu-Arabic numeration system as contrasted with another system may use multiplication examples such as the following:

$$\begin{array}{r} 237 \\ \times\ 12 \\ \hline 474 \\ 237 \\ \hline 2844 \end{array} \qquad \begin{array}{r} \text{CCXXXVII} \\ \text{XII} \\ \hline \text{CCXXXVII} \\ \text{CCXXXVII} \\ \text{MMCCCLXX} \\ \hline \text{MMCCCCCCCLXXXXXXXXVVIIII} = \\ \text{MMDCCCXLIV} \end{array}$$

OPERATIONS ON WHOLE NUMBERS

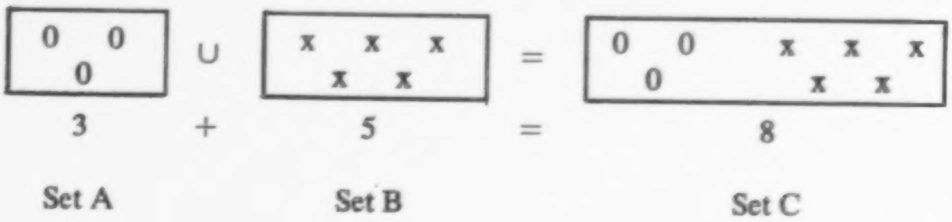
Understanding of the nature of number and the numeration system is necessary if children are to perform operations on numbers efficiently. Such understanding develops gradually as opportunities are provided for children to manipulate objects, discuss and test ideas, experiment with ways to solve problems, and apply principles or generalizations to operations on numbers.

Operations studied in the elementary school are addition and its inverse operation subtraction, and multiplication and its inverse operation division. Helping children to understand these operations and develop skills necessary to use them effectively is a special responsibility of the elementary school.

As operations are performed, certain properties become apparent. For example: The sum of two numbers is not affected by the order in which they are added ($2 + 3 = 3 + 2$); the product of any number and one is the number itself ($8 \times 1 = 8$); the sum of any number and its opposite, or additive inverse, is always zero, ($3 + {}^-3 = 0$). The structure that these properties provide makes it unnecessary to memorize many unrelated facts. Use of the properties is a distinguishing characteristic of contemporary mathematics programs.

Significant as properties are in providing structure for operation on numbers, it should be kept in mind that understanding is developed and the property used before effort is made to name or use vocabulary associated with the various properties.

Algorithms are developed for use in operations on numbers with the idea that there are many ways of working and that a child may find the way that is best suited to him. As understanding of the process develops, most children will find and use efficient patterns.

Mathematical Ideas	Illustrations and Explanations
Addition Addition is the binary operation on two numbers called addends to produce a third number called the sum. The addends are the numbers associated with two disjoint sets. The sum is the number of the set formed by joining or finding the union of the two sets.	From union of disjoint sets children are led to assign the number property to each set and then to perform the operation of addition on the numbers.  In set notation this becomes: N (A) + N (B) = N (C) or a + b = c N (A) is read: "The number property of set A." *a* and *b* are addends and *c* is the sum. Capital letters are used to denote sets and lower case letters are used as variables that represent numbers. Letters have been used in the above illustration, and will be used subsequently throughout the guide, to express generalizations which are true for operations on all numbers.

Mathematical Ideas | **Illustrations and Explanations**

1. PROPERTIES OF ADDITION

Closure: When an operation on two numbers produces a number within the same set of numbers, the set is said to be *closed* under that operation. The set of whole numbers is closed under addition, since the sum is an element of the set.

$3 + 5 = 8$ 3, 5 and 8 are whole numbers. From experience with many similar situations it becomes evident that, for any whole numbers *a* and *b*, $a + b = c$; that *c* is also a whole number and that the set of whole numbers is closed under addition.

Commutative property: The sum of two numbers is not affected by reversing the order of the addends. This applies to addition in all sets of numbers.

$3 + 5 = 8$ and $5 + 3 = 8$ then $3 + 5 = 5 + 3$

In generalizing after much experience it becomes evident that for any two numbers, *x* and *y*: $x + y = y + x$

Associative property: When three or more numbers are addends, the sum is not affected by the way in which the addends are paired. This applies to addition in all sets of numbers.

$$4 + (3 + 2) = 4 + 5 = 9$$

$$\text{and } (4 + 3) + 2 = 7 + 2 = 9$$

$$\text{then } 4 + (3 + 2) = (4 + 3) + 2$$

Therefore, for any numbers, *a*, *b*, and *c*:

$a + (b + c) = (a + b) + c$

Identity element for addition—zero (0): When zero (0) is added to a number, the sum is the number, and zero is called the additive identity.

$6 + 0 = 6$ $0 + 6 = 6$

Therefore, for any number *a*, $a + 0 = a$ or $0 + a = a$

2. ALGORITHMS FOR ADDITION

Any pattern of procedures used to name the result of a mathematical operation is called an algorithm or algorism. Algorithms for each of the operations become more complex as numbers become greater than the base of the numeration system.

Algorithms with two addends less than base: This involves the addition facts shown on the table to the right.

ADDENDS

+	0	1	2	3	4	5	6	7	8	9
0	0									
1	1	2								
2	2	3	4							
3	3	4	5	6						
4	4	5	6	7	8					
5	5	6	7	8	9	10				
6	6	7	8	9	10	11	12			
7	7	8	9	10	11	12	13	14		
8	8	9	10	11	12	13	14	15	16	
9	9	10	11	12	13	14	15	16	17	18

ADDENDS

SUMS

When the commutative property is applied, only 55 facts need to be learned, and by applying the zero property, only 45.

Many algorithms may be developed for any operation. A child should be permitted to use the one which has meaning for him. For example:

$$\begin{aligned} 8+9 &= 8+(2+7) && \text{(Renaming 9 as } 2+7\text{)} \\ &= (8+2)+7 && \text{(Associative property)} \\ &= 10+7 && \text{(Fact previously learned)} \\ &= 17 && \text{(Place value notation)} \end{aligned}$$

Algorithms with one addend less than the base and the other greater than base:

$$\begin{array}{r} 35 \\ +\,2 \\ \hline \end{array} \qquad \begin{array}{rr} 3 \text{ tens } + & 5 \text{ ones} \\ + & 2 \text{ ones} \\ \hline 3 \text{ tens } + & 7 \text{ ones} \end{array} \qquad \begin{array}{r} 35 \\ +\,2 \\ \hline 37 \end{array}$$

$$\begin{array}{r} 94 \\ +\,8 \\ \hline \end{array} \qquad \begin{array}{rr} 9 \text{ tens } + & 4 \text{ ones} \\ + & 8 \text{ ones} \\ \hline 9 \text{ tens } + & 12 \text{ ones} \end{array} \qquad \begin{array}{r} 94 \\ +\,8 \\ \hline 102 \end{array}$$

Renamed as: 10 tens + 2 ones or 102

The expanded notation in these illustrations is used only to develop understanding of the process involved in the algorithm. It should not be used in writing to the point that it becomes needless repetition of an involved process. Some children may go quickly to a shorter algorithm.

Algorithms with addends equal to or greater than base:

$$\begin{array}{r} 15 \\ +\,10 \\ \hline 25 \end{array} \qquad \begin{array}{r} 27 \\ +\,10 \\ \hline 37 \end{array} \qquad \begin{array}{r} 10 \\ +\,36 \\ \hline 46 \end{array} \qquad \begin{array}{r} 10 \\ +\,83 \\ \hline 93 \end{array}$$

$$\begin{array}{r} 26 \\ +\,42 \\ \hline \end{array} \qquad \begin{array}{rl} 2 \text{ tens } + & 6 \text{ ones} \\ 4 \text{ tens } + & 2 \text{ ones} \\ \hline 6 \text{ tens } + & 8 \text{ ones or } 68 \end{array} \qquad \begin{array}{l} 20+6 \\ 40+2 \\ \hline 60+8 \text{ or } 68 \end{array}$$

$$\begin{array}{r} 397 \\ +\,486 \\ \hline \end{array} \qquad \begin{array}{rrr} 3 \text{ hundreds } + & 9 \text{ tens } + & 7 \text{ ones} \\ 4 \text{ hundreds } + & 8 \text{ tens } + & 6 \text{ ones} \\ \hline 7 \text{ hundreds } + & 17 \text{ tens } + & 13 \text{ ones} \end{array}$$

Renamed as: 7 hundreds + 18 tens + 3 ones
Renamed as: 8 hundreds + 8 tens + 3 ones or 883

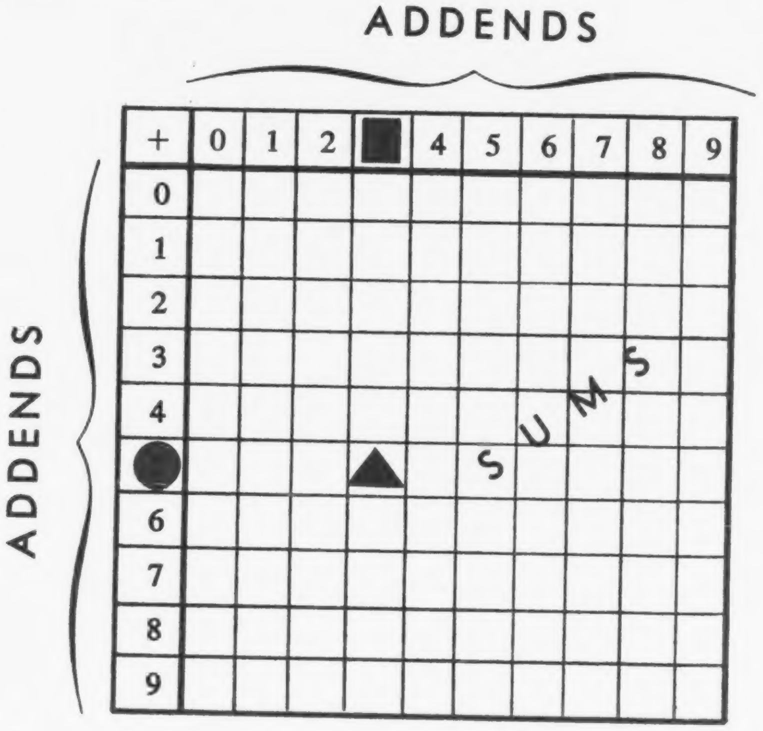
$$\begin{array}{rrr} 300\,+ & 90\,+ & 7 \\ 400\,+ & 80\,+ & 6 \\ \hline 700\,+ & 170\,+ & 13 \end{array} \qquad \begin{array}{r} 397 \\ +\,486 \\ \hline 883 \end{array}$$

$$\begin{aligned} &= 700+(100+70)+(10+3) \\ &= (700+100)+(70+10)+3 \quad \text{associative property} \\ &= 800+80+3=883 \end{aligned}$$

Mathematical Ideas	**Illustrations and Explanations**
Algorithms with three or more addends:	Since addition is a binary operation, both commutative and associative properties are used with emphasis always on the ideas involved rather than on the terminology. $\begin{array}{r} 24 \\ 82 \\ +\underline{57} \end{array}$ $\begin{array}{cc} \text{Tens} & \text{Units} \\ 2 & 4 \\ 8 & 2 \\ \underline{5} & \underline{7} \\ 15 & 13 \end{array}$ Rename as: 1 hundred + (5 + 1) tens + 3 units or 1 hundred + 6 tens + 3 units or 163 $\begin{array}{r} 24 \\ 82 \\ \underline{57} \\ 13 \\ \underline{150} \\ 163 \end{array}$
Subtraction	
Subtraction is the inverse operation of addition. It is the operation of finding one of two addends when their sum and one addend are known.	$7 - 3 = \square$ because $\square + 3 = 7$ $7 - 4 = \square$ because $\square + 4 = 7$
1. PROPERTIES OF SUBTRACTION	
Closure: Since it is not always possible to subtract a whole number from a whole number and get a whole number, the set of whole numbers is not closed under subtraction.	$6 - 2 = 4$, a whole number $4 - 6$ does not name a whole number
Commutative property: Subtraction is not commutative.	$5 - 3 = 2$ $3 - 5 \neq 2$ If $b < a$, $a - b = c$ and $b - a \neq c$
Associative property: Subtraction is not associative.	$(9 - 3) - 2 = 4$ $\quad$ $9 - (3 - 2) = 8$
Zero: Subtracting zero from a number does not change the number.	$8 - 0 = 8$ For any whole number *n* $n - 0 = n$

2. ALGORITHMS FOR SUBTRACTION

With addends less than base: Subtraction facts may be derived from the addition table.



▲ − ● = ■   ■ + ● = ▲

With known addends less than, equal to or greater than the base: Algorithms for subtraction using two or more digit numerals are developed first without renaming and then with renaming.

$$\begin{array}{r} 78\\ \underline{-\ 6}\\ \end{array}\qquad \begin{array}{r} 7\text{ tens} + 8\text{ ones}\\ \underline{-\ (0\text{ tens} + 6\text{ ones})}\\ 7\text{ tens} + 2\text{ ones or } 72 \end{array}\qquad \begin{array}{r} 70 + 8\\ \underline{-\ (0 + 6)}\\ 70 + 2\text{ or } 72 \end{array}\qquad \begin{array}{r} 78\\ \underline{-\ 6}\\ 72 \end{array}$$

$$\begin{array}{r} 78\\ \underline{-\ 10}\\ \end{array}\qquad \begin{array}{r} 7\text{ tens} + 8\text{ ones}\\ \underline{-\ (1\text{ ten} + 0\text{ ones})}\\ 6\text{ tens} + 8\text{ ones or } 68 \end{array}\qquad \begin{array}{r} 70 + 8\\ \underline{-\ (10 + 0)}\\ 60 + 8\text{ or } 68 \end{array}\qquad \begin{array}{r} 78\\ \underline{-\ 10}\\ 68 \end{array}$$

$$\begin{array}{r} 78\\ \underline{-\ 64}\\ \end{array}\qquad \begin{array}{r} 7\text{ tens} + 8\text{ ones}\\ \underline{-\ (6\text{ tens} + 4\text{ ones})}\\ 1\text{ ten} + 4\text{ ones or } 14 \end{array}\qquad \begin{array}{r} 70 + 8\\ \underline{-\ (60 + 4)}\\ 10 + 4\text{ or } 14 \end{array}\qquad \begin{array}{r} 78\\ \underline{-\ 64}\\ 14 \end{array}$$

$$\begin{array}{r} 74\\ \underline{-\ 58}\\ \end{array}\qquad \begin{array}{r} 6\text{ tens} + 14\text{ ones}\\ \underline{-\ (5\text{ tens} + 8\text{ ones})}\\ 1\text{ ten} + 6\text{ ones or } 16 \end{array}\qquad \begin{array}{r} 60 + 14\\ \underline{-\ (50 + 8)}\\ 10 + 6\text{ or } 16 \end{array}\qquad \begin{array}{r} 74\\ \underline{-\ 58}\\ 16 \end{array}$$

$$\begin{array}{r} 374\\ \underline{-\ 98} \end{array}\qquad \begin{array}{r} 300 + 70 + 4\\ \underline{-\ (000 + 90 + 8)} \end{array}\quad \begin{array}{r} \text{Rename the sum as } 300 + 60 + 14\\ \text{then as } 200 + 160 + 14\\ \underline{-\ (000 + 90 + 8)}\\ 200 + 70 + 6\text{ or } 276 \end{array}$$

Mathematical Ideas	Illustrations and Explanations
Multiplication Multiplication is a binary operation on numbers called factors to produce a unique third number called the product. Multiplication of numbers greater than one can be performed using repeated addition of the same whole number. The product is a multiple of each of the factors.	$6 \times 3 = 18$ Factors, 6 and 3, are the numbers being multiplied. 18 is the product, the number obtained through multiplication. ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ $6 + 6 + 6$ ★ ★ ★ ★ ★ ★ (3 addends) ★ ★ ★ ★ ★ ★ $3 + 3 + 3 + 3 + 3 + 3$ ★ ★ ★ (6 addends) ★ ★ ★ ★ ★ ★ ★ ★ ★ For any numbers, $a \times b = c$, *a* and *b* are factors, *c* is the product.
1. PROPERTIES OF MULTIPLICATION	
Closure: The product of two whole numbers is a whole number.	$16 \times 47 = 752$ For any whole numbers, *a* and *b*, $a \times b = c$, and *c* is a unique whole number.
Commutative property: The product of two numbers is not affected by reversing the order of the factors. This applies to multiplication in all sets of numbers.	$6 \times 3 = 18$ $3 \times 6 = 18$ $6 \times 3 = 3 \times 6$ For any numbers, *a* and *b*, $a \times b = b \times a$
Associative property: In finding the product of three or more factors, the way in which the factors are paired does not affect the product.	$(6 \times 3) \times 4 = 72$ $6 \times (3 \times 4) = 72$ $18 \times 4 = 72$ $6 \times 12 = 72$ $(6 \times 3) \times 4 = 6 \times (3 \times 4)$ For any numbers *a*, *b*, and *c*, $(a \times b) \times c = a \times (b \times c)$
Distributive property of multiplication over addition: This property links addition and multiplication in finding the product.	$10 \times 7 = 70$ $10 \times (3 + 4) = (10 \times 3) + (10 \times 4) = 70$ $(6 + 4) \times 7 = (6 \times 7) + (4 \times 7) = 70$ For any numbers *a*, *b*, and *c* $a \times (b + c) = (a \times b) + (a \times c)$ or $(b + c) \times a = (b \times a) + (c \times a)$
Identity element—one: Multiplying by one does not change the number.	$8 \times 1 = 8$ For any number *a*, $a \times 1 = a$ and $1 \times a = a$
Zero: Multiplying any number by zero gives the number zero as product.	$8 \times 0 = 0$ For any number *a*, $a \times 0 = 0$, $0 \times a = 0$

2. ALGORITHMS FOR MULTIPLICATION

Algorithms with two factors less than base:

(This involves the multiplication facts shown on the table to the right.)

Factors

Factors

×	0	1	2	3	4	5	6	7	8	9
0	0	▨	▨	▨	▨	▨	▨	▨	▨	▨
1	0	1	▨	▨	▨	▨	▨	▨	▨	▨
2	0	2	4	▨	▨	▨	▨	▨	▨	▨
3	0	3	6	9	▨	▨	▨	▨	▨	▨
4	0	4	8	12	16	▨	▨	▨	▨	▨
5	0	5	10	15	20	25	▨	▨	▨	▨
6	0	6	12	18	24	30	36	▨	▨	▨
7	0	7	14	21	28	35	42	49	▨	▨
8	0	8	16	24	32	40	48	56	64	▨
9	0	9	18	27	36	45	54	63	72	81

When the commutative property and the property of zero are applied, it can be observed that facts in the shaded portion of the chart are the same as facts in the unshaded section.

The distributive property may be used to arrive at facts with factors 6 through 9.

If $5 \times 8 = 40$ is known, 7×8 may be found:

$$\begin{aligned} 7 \times 8 &= (5 + 2) \times 8 \\ &= (5 \times 8) + (2 \times 8) \\ &= 40 + 16 \\ &= 56 \end{aligned} \qquad \text{or} \qquad \begin{aligned} 7 &\times (4 + 4) \\ &= (7 \times 4) + (7 \times 4) \\ &= 28 + 28 \\ &= 56 \end{aligned}$$

Algorithms with one factor equal to the base:

$1 \times 10 = 10$
$2 \times 10 = 20$
$3 \times 10 = 30$

$27 \times 10 = 270$
.
.
.

$9 \times 10 = 90$
$10 \times 10 = 100$
$11 \times 10 = 110$

$115 \times 10 = 1{,}150$
.
.
.

$15 \times 10 = 150$
.
.
.

Algorithms with one factor greater than base:

without renaming

$3 \times 23 = 69$

$$\begin{array}{rl} 23 & \\ \times\ 3 & \\ \hline 9 & (3 \times \ 3 = \ 9) \\ 60 & (3 \times 20 = 60) \\ \hline 69 & \end{array} \qquad \begin{array}{r} 23 \\ \times\ 3 \\ \hline 69 \end{array} \qquad \begin{array}{r} 3 \\ \times\ 23 \\ \hline 9 \\ 60 \\ \hline 69 \end{array}$$

with renaming

$$\begin{array}{rcl} 23 = & & 2 \text{ tens} + \ 3 \text{ ones} \\ \underline{\times\ \ 4} = & \times & \underline{\qquad\qquad\quad 4 \text{ ones}} \\ & & 8 \text{ tens} + 12 \text{ ones} = \\ & & 8 \text{ tens} + \ 1 \text{ ten} + 2 \text{ ones} = \\ & & 9 \text{ tens} + \ 2 \text{ ones} = 92 \end{array}$$

$$\begin{aligned}4 \times 23 &= 4 \times (20+3) && \text{Renaming 23 as } (20+3)\\ &= (4 \times 20) + (4 \times 3) && \text{Distributive property}\\ &= 80 + 12 && \text{Multiplication}\\ &= 80 + (10+2) && \text{Renaming 12 as } (10+2)\\ &= (80+10)+2 && \text{Associative property}\\ &= 90 + 2 && \text{Addition}\\ &= 92 && \text{Addition}\end{aligned}$$

$$\begin{array}{r} 23\\ \times 4\\ \hline 12\\ 80\\ \hline 92\end{array} \quad = \quad \begin{array}{r} 20 + 3\\ \times \quad 4\\ \hline 80 + 12 \text{ or } 92\end{array} \quad \begin{array}{r} 23\\ \times 4\\ \hline 92\end{array}$$

Algorithms with two factors greater than base:

$$\begin{aligned}23 \times 46 &= (20+3) \times (40+6) && \text{Renaming 23 as } (20+3) \text{ and 46 as } (40+6)\\ &= 20 \times (40+6) + 3 \times (40+6) && \text{Distributive property}\\ &= (20 \times 40) + (20 \times 6) + (3 \times 40) + (3 \times 6) && \text{Distributive property}\\ &= 800 + 120 + 120 + 18 && \text{Multiplication}\\ &= 920 + 138 && \text{Addition}\\ &= 1{,}058 && \text{Addition}\end{aligned}$$

$$\begin{array}{rl} 46 & \\ \times 23 & \\ \hline 18 & = (3 \times 6)\\ 120 & (3 \times 40)\\ 120 & (20 \times 6)\\ 800 & (20 \times 40)\\ \hline 1{,}058 & \end{array} \qquad\qquad \begin{array}{r} 46\\ \times 23\\ \hline 138\\ 92\\ \hline 1{,}058\end{array}$$

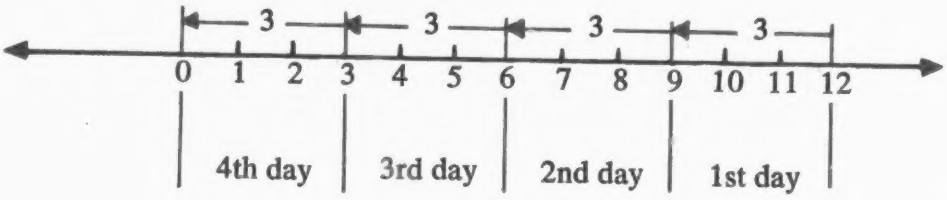
Division

Division is the inverse operation of multiplication. It is the operation of finding a factor when the product and one factor are known. The unknown factor may be called the quotient, the known factor the divisor, and the product may be called the dividend. Division is the equivalent of repeated subtraction of the same whole number.

$138 \div 6 = \square$ implies $6 \times \square = 138$
$138 \div 23 = \square$ implies $\square \times 23 = 138$

For any natural numbers a and b, if $a \times b = c$ with c a whole number, then $a = c \div b$ and $b = c \div a$

$$\begin{array}{r|l} 4\overline{)12} & \\ -4 & 1\\ \hline 8 & \\ -4 & 1\\ \hline 4 & \\ -4 & +1\\ \hline 0 & 3\end{array}$$

Mathematical Ideas	Illustrations and Explanations
Zero in division: Division is considered as the inverse of multiplication. Zero as a divisor is meaningless and division by zero is not possible in the set of whole numbers.	$15 \div 5 = 3$ $3 \times 5 = 15$ $0 \div 6 = 0$ $0 \times 6 = 0$ $6 \div 0$ is meaningless because any number times zero is zero, never 6. Zero cannot be used as a divisor.
NEVER DIVIDE BY ZERO!	
Partitive application: Division can be used to find the number of members in each of *n* equivalent sets.	Twenty-four girls are to be divided into four teams with the same number of girls on each team. How many girls will be on each team? $24 \div 4 = 6$ Six girls on a team.
Measurement application: Division can be used to find the number of equivalent sets of *n* each in a given set.	Each day a child uses three pages from a notebook. If the book contains twelve pages, how many days will it last? $12 - 3 = 9$ pages left after first day $9 - 3 = 6$ pages left after second day $6 - 3 = 3$ pages left after third day $3 - 3 = 0$ pages left after fourth day or $12 \div 3 = 4$ The book will last for four days.  3 3 3 3 0 1 2 3 4 5 6 7 8 9 10 11 12 4th day 3rd day 2nd day 1st day
1. **Properties of Division**	
Closure: Since it is not always possible to divide a whole number by a whole number and get a whole number, the set of whole numbers is not closed under division.	$6 \div 3 = 2$ a whole number $3 \div 6 \neq$ a whole number
Commutative property: Division is not commutative.	$12 \div 4 = 3$ $4 \div 12 \neq 3$
Associative property: Division is not associative.	$(18 \div 2) \div 3 = 3$ $18 \div (2 \div 3) \neq 3$
Distributive property: Division is distributive with respect to addition except that the distribution must be over the dividend with each operation producing a whole number.	$369 \div 3 = (300 + 60 + 9) \div 3$ $= (300 \div 3) + (60 \div 3) + (9 \div 3)$ $= 100 + 20 + 3$ $= 123$ $(a + b + c) \div d = (a \div d) + (b \div d) + (c \div d)$

2. ALGORITHMS FOR DIVISION

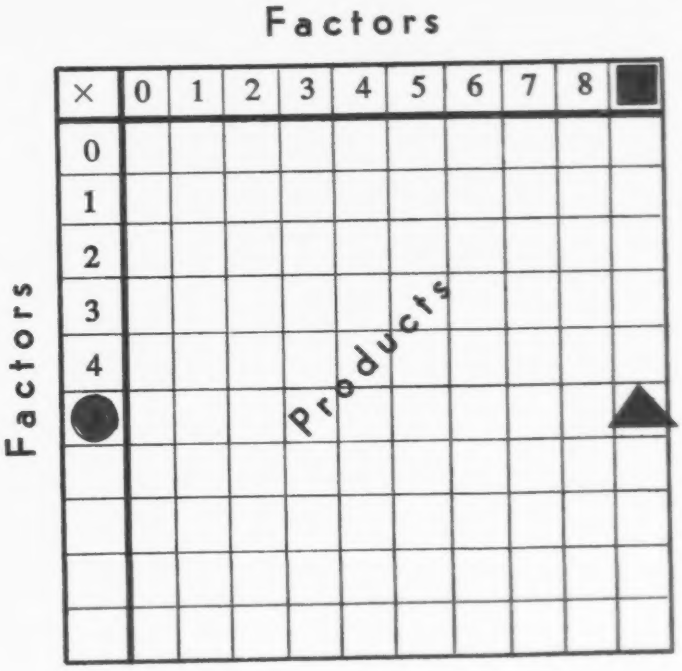
Algorithms with factors less than base: This involves facts learned from the multiplication.

45 ÷ 9 = ☐ implies ☐ × 9 = 45

$$\begin{array}{r} 5 \\ 9\overline{)45} \end{array}$$

▲ ÷ ■ = ●

● × ■ = ▲



The operation of division can be carried out with zero remainders or non-zero remainders.

37 ÷ 5 = ☐ implies ☐ × 5 = 37

☐ × 5 = 37
7 × 5 = 35
37 − 35 = 2 remainder
(7 × 5) + 2 = 37

$$\begin{array}{r} 7 \\ 5\overline{)37} \\ \underline{35} \\ 2 \end{array} \qquad \begin{array}{r} q \\ b\overline{)a} \\ \underline{} \\ r \end{array}$$

a = (b × q) + r

b = divisor, known factor
a = dividend, product
q = quotient, unknown factor
r = remainder

Mathematical Ideas | **Illustrations and Explanations**

Algorithms with one factor equal to base:

10 ÷ 10 = 1 150 ÷ 10 = 15

20 ÷ 10 = 2 3,460 ÷ 10 = 346

Algorithms with factors greater than base: With one or both factors greater than base the algorithm may be developed in several ways.

$$
\begin{array}{r|rr}
7\overline{)959} & & \\
-\underline{700} & 100 & (7 \times 100 = 700) \\
259 & & \\
-\underline{210} & 30 & (7 \times 30 = 210) \\
49 & & \\
-49 & \underline{+7} & (7 \times 7 = \underline{49}) \\
\hline
0 & 137 & 959
\end{array}
\qquad
\begin{array}{r}
\underline{137} \\
7 \\
30 \\
100 \\
7\overline{)959} \\
\underline{700} \\
259 \\
\underline{210} \\
\underline{49} \\
49
\end{array}
\qquad
\begin{array}{r}
137 \\
7\overline{)959} \\
\underline{7} \\
25 \\
\underline{21} \\
\underline{49} \\
49
\end{array}
$$

959 ÷ 7 = ☐ implies ☐ × 7 = 959

959 ÷ 7 = 137 implies 137 × 7 = 959

$$
\begin{array}{r}
\underline{53} \\
3 \\
50 \\
7\overline{)374} \\
\underline{350} \\
24 \\
\underline{21} \\
3 \text{ remainder}
\end{array}
\qquad
\begin{array}{r}
\left.\begin{array}{r} 3 \\ 50 \\ 100 \end{array}\right\} = 153 \\
24\overline{)3678} \\
\underline{2400} \\
1278 \\
\underline{1200} \\
78 \\
\underline{72} \\
6 \text{ remainder}
\end{array}
\qquad
\begin{array}{r}
153 \\
24\overline{)3678} \\
\underline{24} \\
127 \\
\underline{120} \\
78 \\
\underline{72} \\
6 \text{ remainder}
\end{array}
$$

Any problem with two known elements can be solved if it can be correctly stated in either of the following forms:

addend + addend = sum
factor × factor = product

Subtraction is the operation of finding the unknown addend. Division is the operation of finding the unknown factor when the remainder is zero.

3. SQUARE ROOT OF A NUMBER

The square root of a number is one of two equal factors whose product is the number. The symbol, $\sqrt{\ }$, designates the square root of a number.

Algorithm for finding the square root of a number: Unike the other operations that have been encountered, all of which are binary, the operation of finding the square root of a number is *unary* in that it involves only one number.

The algorithm for finding the square root of a number may be described as a "divide and average" procedure. The goal is to get the divisor and quotient to be the same or as close as you wish.

1. Estimate the value of the square root.
2. Using the estimate as a divisor, divide the number to find the quotient.
3. Average the divisor and quotient. The result is a closer approximation to the square root of the number.
4. Using this new approximation as a divisor, repeat steps 2 and 3 to obtain an approximation of the square root as correct as required. However, with practice, two to three repetitions should produce a square root as correct as necessary for most purposes.

Example:

$\sqrt{4} = 2$ because $2 \times 2 = 4$

$\sqrt{25} = 5$ because $5 \times 5 = 25$

$\sqrt{\frac{9}{16}} = \frac{3}{4}$ because $\frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$

$\sqrt{2.25} = 1.5$ because $1.5 \times 1.5 = 2.25$

Since $3 \times 3 = 9$ and $(-3) \times (-3) = 9$, it is agreed that

$\sqrt{9} = 3$ and $-\sqrt{9} = -3$.

Example: Find $\sqrt{19}$

1. Since $4 \times 4 = 16$, try 4 as a first estimate of $\sqrt{19}$.

2. $19 \div 4 = 4.75$

$$\begin{array}{r} 4.75\\ 4\overline{)19.00}\\ \underline{16}\\ 30\\ \underline{28}\\ 20\\ \underline{20} \end{array}$$

3. $\sqrt{19}$ lies between 4 and 4.75 $\quad \frac{4 + 4.75}{2} = 4.375$

4. Repeat step 3, averaging 4.375 divisor and 4.3429 (quotient)

$$\frac{4.375 + 4.343}{2} = 4.359$$

$\sqrt{19} \sim 4.359$

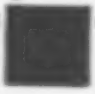
TEACHING SUGGESTIONS FOR OPERATIONS ON WHOLE NUMBERS

Addition

Readiness for introducing operation of addition is provided when disjoint sets are joined, i.e., by making the union of the two sets. After children have many experiences with joining sets, the teacher may introduce the operation of addition through the following steps:

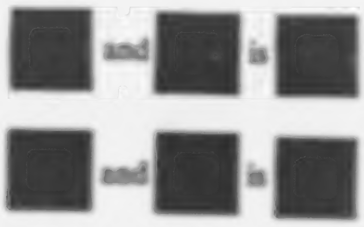
1. Demonstrating the joining of a set of four objects and a set of one object to form a set of five objects.
2. Explaining that joining sets of objects helps in thinking about adding numbers. The addition of numbers is stated as "four plus one equals five."
3. Writing 4 + 1 = 5 on chalkboard and repeating "four plus one equals five."
4. Isolating + and = and explaining again their names and meanings.
5. Using other examples as needed in group instruction then repeating this development as children individually join sets of objects.

Children can discover answers to number facts or find other names for numbers, (Example: 5 = 4 + 1, 5 = 3 + 2, or 5 = 0 + 5) by manipulating sets of objects, or using such manipulative devices as a bead frame, beads on a wire, colored plastic clothespins attached to a coat hanger. The flannel board, magnetic board, and dot cards can also be used for this purpose.



Children may draw dominoes to illustrate number sentences.

They may make cards showing numerals and also cards showing sets of dots. Use of these can vary in many ways.



Many opportunities for problem solving are available following the introduction of the operation of addition. Primary children may:

1. Dramatize an addition situation and then write a mathematical sentence.
2. Discuss a word problem and get from it mathematical ideas which are then transferred to a mathematical sentence.
3. Solve the sentence and relate the result to the problem situation.
4. Make up other problems for classmates to solve.

1. PROPERTIES OF ADDITION

Closure: Through discussion the idea of closure can be developed deductively. Number facts may be listed on the chalkboard and children asked to supply the sums. (This may also be done using addends of two and three digit numerals.) Such questions as the following may aid in developing the idea of closure:

a. To what set of numbers do the addends belong?
b. To what set of numbers do the sums belong?
c. Is there an example when the sum of two whole numbers is not a whole number?
d. What can always be said about the sum when the addends are whole numbers?

Understanding of the property provides a quick check for computation. When odd and even numbers are understood, a similar discussion can be carried on to discover that the sum of even numbers is always an even number. When odd numbers are treated in the same way, an additional question must be asked: What is true about the sum when even numbers are added that is not true for the sum of odd numbers? Charts similar to the following to illustrate the idea may be prepared by children:

+	O	E
O	E	O
E	O	E

While young children are able to gain understanding of closure, they are not expected to verbalize the property at an early age.

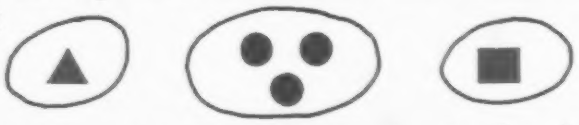
Commutative property: Joining sets of objects can help children discover the commutative property since it is easy to see that order is not important; that the result is the same regardless of how the sets are joined together. As children understand that the order in which sets of objects are joined does not affect the sum, pairs of addends may be written on the chalkboard as follows:

$2 + 3 = \square$ $\qquad$ $3 + 2 = \square$

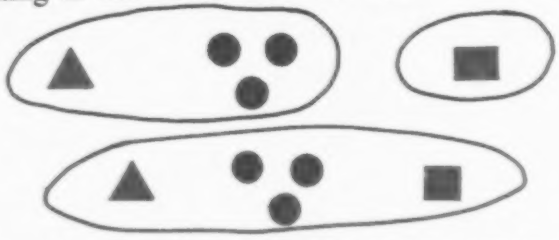
$$\begin{array}{r}2\\+3\\\hline\square\end{array}\qquad\qquad\begin{array}{r}3\\+2\\\hline\square\end{array}$$

As sums are recorded and the process repeated with other pairs of addends, children should be able to express in their own words the generalization that the order in which the operation of addition is performed on two addends does not affect the sum. It is not necessary that the term "commutative" be used with young children since it is the idea and not the terminology which is important.

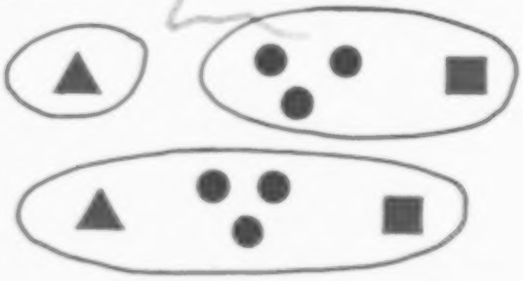
Associative property: Three sets of objects may be used to develop understanding of the associative property:



a. By joining the first and second sets, then by joining to this union the third set.



b. By using the same sets, joining second and third sets, then to this union joining the first set.



Children can observe that the result is the same regardless of how the sets are grouped. They need many experiences in joining sets in this way. Pictures of objects can also be used with children circling the order in which grouping is done.

When understanding of the property has been developed through manipulation of concrete objects, more abstract ideas may be introduced by:

a. Writing mathematical sentences on the chalkboard: $3 + 2 + 4 = \square$ $\quad$ $3 + 2 + 4 = \square$

b. Reviewing the idea that only two numbers may be added at one time.

c. Eliciting from children ways in which the sum might be found.

d. Putting parentheses around pairs of numbers as groupings are suggested:

$(3 + 2) + 4 = 3 + (2 + 4)$ and calling attention to this use of parentheses.

e. Completing the addition process and comparing sums.

f. Following this procedure with other examples as both group and individual activity.

g. Providing opportunity for children to generalize the property in their own words.

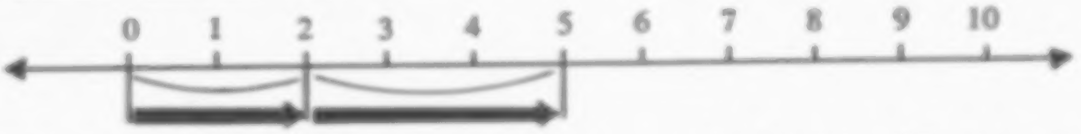
Identity element: Activities similar to those described above may be used to develop understanding of zero as the identity element for addition. By using the empty set when concrete objects are manipulated and zero when writing mathematical sentences, children can be led to see that with zero as one addend the sum is the number of the other addend.

$3 + 0 = 3$ $\qquad$ $0 + 3 = 3$

2. ALGORITHMS FOR ADDITION

Algorithms with two addends less than base: After the number facts are discovered, either through activities with concrete objects, through use of the number line, or through renaming and the application of the associative property, they should be learned. As motivation for learning, usefulness of the commutative property in reducing the number of facts to be learned may be capitalized on. See table page 31.

The number line may also be used in discovering sums for basic facts.



$2 + 3 = 5$

Such games as the following may be used as drill in learning number facts:

With a set of small objects in each hand, a child writes the sum of the sets on the chalkboard. Classmates take turns in guessing the number of objects in each hand. The correct guess determines the next leader.

Algorithms with one addend less and one greater than base: As when developing understanding of place value children may manipulate bundles of ten sticks and single sticks to gain understanding of the process involved in the algorithm. For example, to find the sum of 45 and 3, they may join a set of four bundles of ten and five single sticks to a set of three single sticks and record the results as

$$\begin{array}{lr} & \text{4 tens and 5 ones} \\ + & \text{3 ones} \\ \hline & \text{4 tens and 8 ones.} \end{array}$$

Most children will advance quickly from the "concrete stage" to the ability to do similar computations using:

Word names	Expanded notation	Conventional algorithm
4 tens and 5 ones	40 + 5	45
3 ones	3	+ 3
4 tens and 8 ones	40 + 8 = 48	48

Algorithms with addends equal to or greater than base: This algorithm may also be introduced through the manipulation of bundles of sticks. For example, to find the sum of 97 + 86, the procedure of joining sets of bundles and single sticks is followed as outlined above. Through appropriate questions children are led to see the need to regroup the thirteen single sticks into one bundle of ten and three ones. The procedure may be recorded again as above using:

Word names	Expanded notation	Conventional algorithm
9 tens and 7 ones	90 + 7	97
8 tens and 6 ones	80 + 6	+ 86
17 tens and 13 ones = 17 tens + 1 ten + 3 ones	170 + 13 = 170 + 10 + 3	183
= 18 tens + 3 ones	= 180 + 3	
	= 183	

Children may develop different algorithms as they gain understanding and skill that leads to the shorter more efficient algorithms. For example:

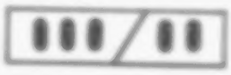
$$\begin{array}{r} 94 \\ +8 \\ \hline 90 \\ 12 \\ \hline 102 \end{array} \qquad \begin{array}{r} 94 \\ +8 \\ \hline 12 \\ 90 \\ \hline 102 \end{array} \qquad \begin{array}{r} 94 \\ +8 \\ \hline 102 \end{array}$$

Each child should be able to explain why numerals belong in the various places.

Subtraction

After children have had many experiences separating sets of objects into subsets, the operation of subtraction may be introduced. Beginning with a set of five objects, the teacher moves one object slightly to the side, being sure that both subsets are in full view of children; then asks for the number of the remaining set. It should be clearly explained that separating sets helps with understanding of subtraction and that the subtraction is stated as "five minus one equals four." It is then written on the chalkboard as 5 − 1 = 4 and read as "five minus one equals four." An illustration with concrete objects and discussion of the fact that 5 represents the sum or whole set and that 1 and 4 name the parts or subsets is necessary. Children will be familiar with the equality sign from work with addition but minus (−) as the sign that means to subtract must be completely understood.

Using other examples children should manipulate sets of counters at their desks with the teacher writing mathematical sentence each time on the chalkboard. When this process is understood, a set of objects may be displayed and separated in some such manner as the following:



Children are asked to give the mathematical sentence for the subtraction of two from five (5 − 2 = 3). Demonstrating with objects the teacher can lead children to see that another way to write the sentence might be 3 + □ = 5 since 5 represents the sum or whole set. This sentence then asks "3 and what make 5?"

Since they already know that 3 + 2 = 5, demonstrations with objects and questions should help them to see that 5 − 3 = 2 and that □ + 3 = 5 is another way of writing the same mathematical idea.

The inverse relationship of addition and subtraction may be further emphasized by:

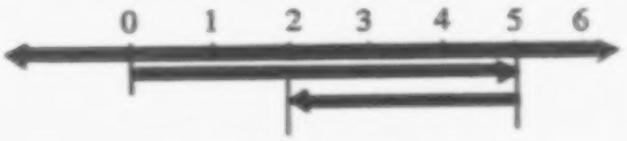
1. Displaying a set of five objects
2. Separating them into sets of three and two

3. Asking for mathematical sentences that this picture brings to mind
4. Writing the sentences on the chalkboard:

$3+2=5 \qquad 5-3=2$

$2+3=5 \qquad 5-2=3$

Much work with sets of objects may be necessary before all children are able to give the four sentences each time. Domino cards may be used to strengthen the understanding and the number line may be used in the discovery of these and other subtraction facts.



1. Properties of Subtraction

Closure: The closure property for subtraction should be developed very informally since children are not expected to verbalize the property. They can realize that in the set of whole numbers it is not always possible to subtract. By separating sets of objects and by looking at mathematical sentences such as: $7 - 2 =$, $2 - 4 =$, $5 - 3 =$, $5 - 7 =$, $3 - 5 =$, $9 - 11 =$, children will become aware that it is not always possible to subtract using the set of whole numbers.

Commutativity: Children may likewise manipulate concrete objects to see that the order in which the sum and the known addend are arranged is important in subtraction:

$$5 - 2 \neq 2 - 5 \qquad 6 - 4 \neq 4 - 6.$$

Although children are not expected to verbalize the fact that subtraction is not commutative, they see, by manipulating objects, that unlike answers are obtained when numbers to be subtracted are in a different order. They understand that order *does* affect the result in subtraction.

Identity element: Children who discover that removing an empty set from a set of objects has no effect on the number of the set will be able to demonstrate, as well as understand, that subtraction of zero from a number does not change the number.

2. Algorithms for Subtraction

With addends less than base: Children who are familiar with the table of addition facts, and who understand the inverse relationship of addition and subtraction as involved in mathematical sentences such as $4 + \square = 6$ will quickly see how subtraction facts may be derived from the addition table. They should be led to see that, if they know addition facts, they can subtract.

With known addends less than, equal to or greater than base: Activities during development of ideas of place value provide readiness for renaming needed in subtraction. Children can bundle and rebundle sticks to find different names for numbers. For example, use bundled sticks to show that:

52 = 5 tens, 2 ones
= 4 tens, 12 ones

352 = 3 hundreds, 5 tens, 2 ones
= 3 hundreds, 4 tens, 12 ones
= 2 hundreds, 15 tens, 2 ones

4 hundreds, 6 tens, 7 ones = 467
3 hundreds, 16 tens, 7 ones = 467
3 hundreds, 15 tens, 17 ones = 467

Separating sets of sticks arranged in bundles of ones and tens illustrates the meaning of the subtraction algorithm. As sticks are separated, the operation can be recorded on the chalkboard using word names. For example:

Renamed

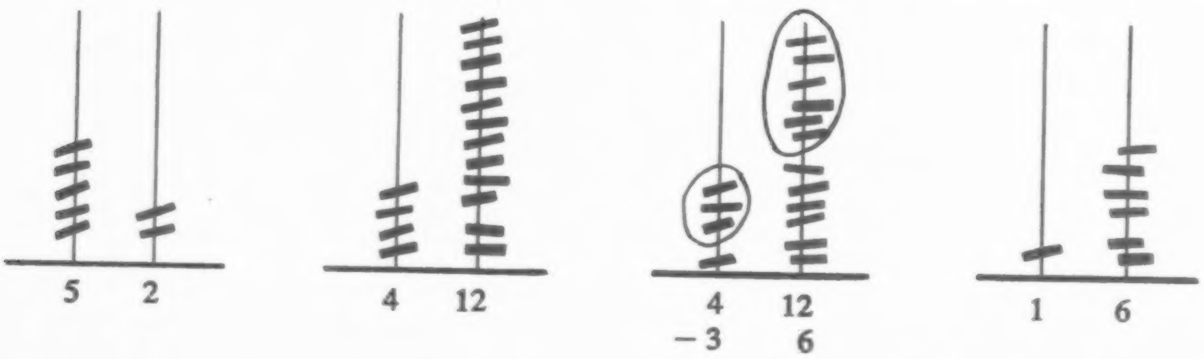
$$\begin{array}{r} \text{5 tens, 2 ones} \\ -\text{ 3 tens, 6 ones} \\ \hline \end{array} \qquad \begin{array}{r} \text{4 tens, 12 ones} \\ -\text{ 3 tens, 6 ones} \\ \hline \text{1 ten, 6 ones or 16} \end{array}$$

Then expanded notation

$$\begin{array}{r} 50 + 2 \\ -30 + 6 \\ \hline \end{array} \qquad \text{Renamed} \qquad \begin{array}{r} 40 + 12 \\ -30 + 6 \\ \hline 10 + 6 = 16 \end{array}$$

Because of individual differences, it is important to note that particularly in subtraction which requires renaming, some children will need to stay on the concrete level longer while other children, building on prior experience, will readily move to word names and expanded notation forms. The goal is to be able to use the conventional algorithm with understanding so each child will get to this stage by following his own individual pattern.

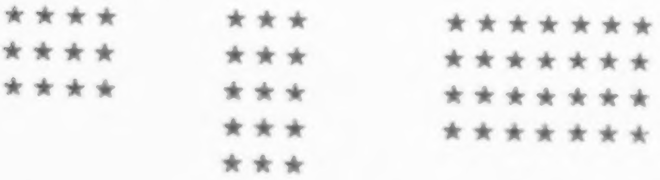
The abacus is also a useful device in developing the subtraction algorithm.



$$\begin{array}{r} 52 \\ -\,36 \\ \hline 16 \end{array}$$

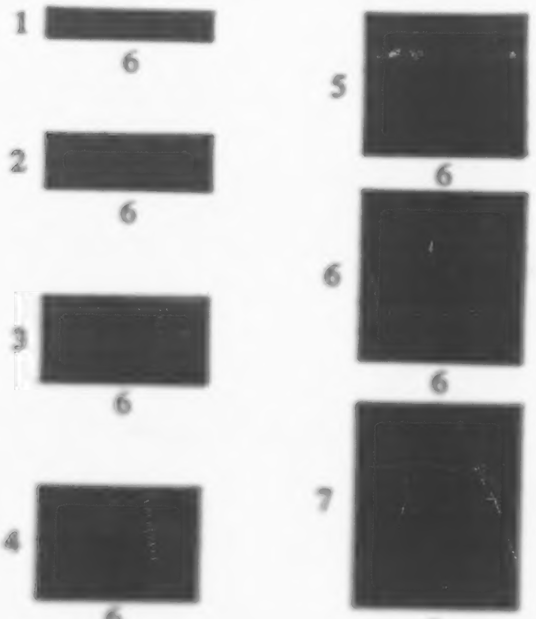
Multiplication

By manipulating equivalent sets of objects and preparing and using rectangular arrays children can discover products.

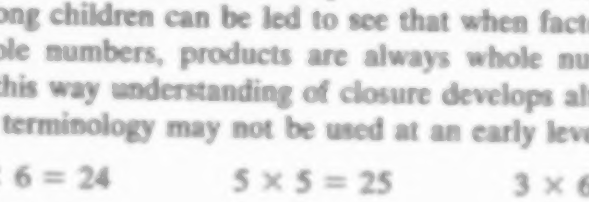


Large demonstration arrays and peg boards on which arrays of different sizes can be arranged may also prove useful in developing understanding.

It may be helpful for children to cut squared paper into strips representing arrays of increasing size to aid in learning multiplication facts. For example:



Experience with such concrete and semi-concrete materials as those mentioned above leads to understanding that the product for factors greater than one can be found by repeated addition of the same number. Also the different factorizations of one number may be illustrated with one set of blocks, squares, or other objects. For example:

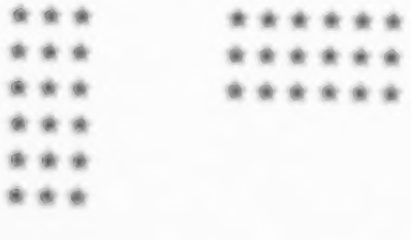


1. PROPERTIES OF MULTIPLICATION

Closure: By discussing the sets of numbers to which factors and products in multiplication of whole numbers belong children can be led to see that when factors are whole numbers, products are always whole numbers. In this way understanding of closure develops although the terminology may not be used at an early level.

$4 \times 6 = 24$ $\quad 5 \times 5 = 25$ $\quad 3 \times 6 = 18$

Commutative property: By rotating a rectangular array, thus reversing rows and columns, children can gain understanding of the commutative property of multiplication. They can readily see that reversing the factors does not change the product.



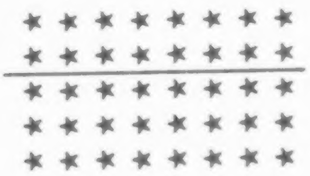
$3 \times 6 = 6 \times 3$

Associative property: Much previous experience with concrete and semi-concrete materials and with properties of addition should enable children to understand readily that the way in which factors are grouped for the binary operation of multiplication does not affect the sum. Opportunity to experiment with the grouping of factors in multiplication should help all children to recognize that:

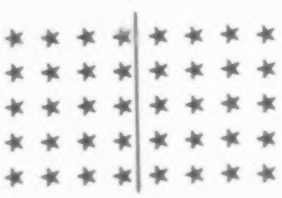
$$(2 \times 3) \times 4 = 2 \times (3 \times 4)$$

$$(5 \times 8) \times 6 = 5 \times (8 \times 6)$$

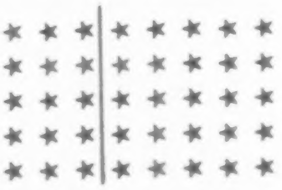
Distributive property of multiplication over addition: By folding a paper showing an array, children can discover how the distributive property of multiplication over addition works.



$$\begin{aligned} 5 \times 8 &= (2+3) \times 8 \\ &= (2 \times 8) + (3 \times 8) \\ &= 16 + 24 \\ &= 40 \end{aligned}$$



$$\begin{aligned} 5 \times 8 &= 5 \times (4+4) \\ &= (5 \times 4) + (5 \times 4) \\ &= 20 + 20 \\ &= 40 \end{aligned}$$



$$\begin{aligned} 5 \times 8 &= 5 \times (3+5) \\ &= (5 \times 3) + (5 \times 5) \\ &= 15 + 25 \\ &= 40 \end{aligned}$$

After understanding has been developed, the distributive property may be used in both written and mental computation. For example: It may be used in discovering the more difficult multiplication facts:

$$\begin{aligned} 7 \times 9 &= (5+2) \times 9 \\ &= (5 \times 9) + (2 \times 9) \\ &= 45 + 18 \\ &= 50 + 13 \\ &= 63 \end{aligned}$$

$$\begin{aligned} 7 \times 9 &= 7 \times (4+5) \\ &= (7 \times 4) + (7 \times 5) \\ &= 28 + 35 \\ &= 63 \end{aligned}$$

It is also involved in multiplication algorithms:

$$\begin{array}{r} 765 \\ \times\ 32 \\ \hline 1{,}530 \\ 22{,}950 \\ \hline 24{,}480 \end{array}$$

765 × 32 renamed and multiplied as (2 + 30) × 765 = (2 × 765) + (30 × 765) = 1,530 + 22,950

1,530 + 22,950 added = 24,480

Identity element—one: By working several examples where one is a factor, children can readily see that the product is always the same as the other factor, ($1 \times 7 = 7; 9 \times 1 = 9$) and that one is a factor of every whole number.

Zero: The union of empty sets, multiplication as repeated addition, demonstrates that multiplication where one factor is zero gives a product of zero.

$$N(A) = 0; \quad N(B) = 0; \quad N(C) = 0; \quad N(D) = 0$$

$$A \cup B \cup C \cup D = \emptyset \quad \text{Hence } 4 \times 0 = 0.$$

This experience should be repeated if children persist with errors in algorithms.

2. ALGORITHMS FOR MULTIPLICATION

Algorithms with two factors less than base: See Suggestions on page 46 for development of number facts. Once children understand the operation of multiplication as repeated addition of the same number and know how to discover products for the basic facts, they should learn the facts.

Discovering patterns in the chart of basic multiplication facts motivates learning, provides practice needed for learning facts and stimulates observation. For example: By referring to multiplication chart similar to the one on page 36 children may be stimulated to discover patterns by such questions as the following:

a. What can you say about all products under the column headed by two, by three, by four, etc.?

b. What patterns do you see formed by units digits in products in the column under even factors?

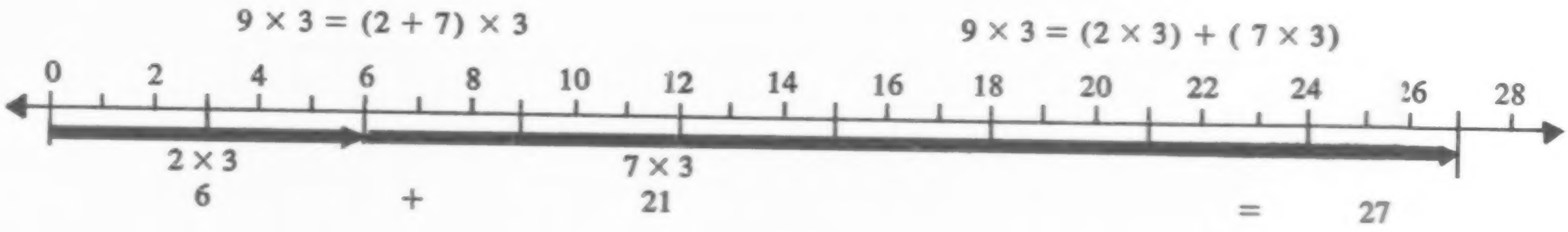
c. Is this true for products in columns under odd factors?

d. What pattern results when digits of products are added?

Games can be used to motivate the learning of basic facts. Children can invent games of their own.

See also page 47 for suggestions regarding the use of distributive property to arrive at facts with factors 6 through 9. This involves renaming one factor, then applying the distributive property and utilizing facts previously learned.

The number line may also be used to discover answers to basic facts.

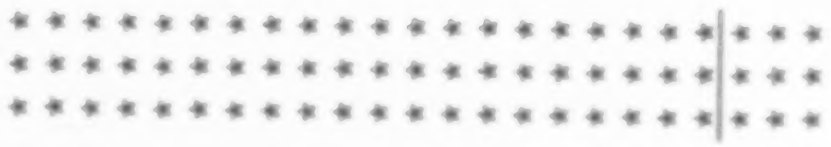


Algorithms with one factor equal to base: Using dimes the pattern of multiplying by the base (10) becomes apparent:

1 dime	1 × 10 = 10	1 dime is 10 cents.
2 dimes	2 × 10 = 20	2 dimes are 20 cents.
3 dimes	3 × 10 = 30	3 dimes are 30 cents.
.	.	.
.	.	.
.	.	.

Algorithms with one factor greater than base: The distributive property of multiplication over addition as developed on page 47 produces the pattern or the multiplication algorithm. For example:

$$\begin{array}{r} 23 \\ \underline{\times 3} \end{array}$$



$$\begin{aligned} (20 + 3) \times 3 &= (20 \times 3) + (3 \times 3) \\ &= 60 + 9 \\ &= 69 \end{aligned} \qquad \begin{array}{r} 23 \\ \underline{\times 3} \\ 69 \end{array}$$

Algorithms with two factors greater than base: Ideas suggested for algorithms with one factor greater than base may be applied to develop understanding of algorithms with two factors greater than base; although most children should not need to rely heavily upon a physical model. Ideas presented on page 37 should be helpful in developing and recording the process. Although some children may need to continue to work at the developmental level, most should be encouraged to use the conventional algorithm.

Division

Division may be presented as the inverse ofmultiplication by asking questions to make the relationship clear. For example:

Five times what equals fifteen?

What times seven equals twenty-one?

After answering many such questions children can see that when two factors are known, they multiply to find the product and when the product and one factor are known they divide to find the missing factor.

☐ × 3 = 15 15 ÷ 3 = ☐

Zero in division: Special attention should be given to the fact that division by zero is meaningless. The idea that zero times any number is zero should be reviewed. The mathematical sentences 6 ÷ 0 = ☐ and ☐ × 0 = 6 may be written on the chalkboard, and followed by such questions as:

What times zero equals six? ☐ × 0 = 6

Is there any number that answers the question? (NO)

Also 0 ÷ 0 = ☐ implies ☐ × 0 = 0
What number or numbers makes the statement true? (All numbers)

If this procedure is followed by other examples and similar questions used, children will realize that division by zero is meaningless.

Partative application: Children can gain understanding of the partative application of division by using counters to find solutions to such problem situations as:

a. Determining each person's share when 24 pieces of candy are divided among 4 children

b. Determining how many children will ride in each car when 6 cars are used to take 24 children to the zoo.

Measurement application: Understanding of the measurement application of division may also be developed by the use of counters or pieces of string in the solution of such problem situations as the following:

a. Determining how many children may be served 2 hot dogs from a box of 36

b. Determining how many 6 inch pieces may be cut from 48 inches of ribbon.

1. PROPERTIES OF DIVISION

Closure: Division examples similar to the following may be written on the chalkboard and discussed:

$4\overline{)16}$ $5\overline{)25}$ $5\overline{)26}$ $16\overline{)8}$

$5\overline{)3}$

Children will observe, as quotients are obtained, that it is not always possible to divide using the set of whole numbers. It should be kept in mind that understanding of the idea is more important than verbalization of the property.

Commutative property: Mathematical sentences such as the following may be written on the chalkboard, solved, and discussed:

$12 \div 3 = 4$ $3 \div 12 \neq 4$

$18 \div 6 = 3$ $6 \div 18 \neq 3$

Children can see that division is not commutative. The way in which divisor and dividend are arranged does affect the quotient.

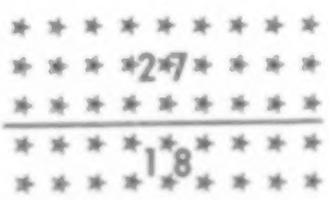
Associative property: Mathematical sentences similar to the following may be written on the chalkboard, solved, and discussed:

$18 \div 6 \div 3 =$ $(18 \div 6) \div 3 = 3 \div 3 = 1$ $18 \div (6 \div 3) = 18 \div 2 = 9$

After experience with a number of similar mathematical sentences, children can see that the associative property does not hold for division. The order in which the divisions are performed does affect the quotient.

Distributive property: An array may be folded to illustrate the distributive property. For example:

$$\begin{aligned} 45 \div 9 &= \\ (27 + 18) \div 9 &= \\ (27 \div 9) + (18 \div 9) &= \\ 3 + 2 &= 5 \\ 45 \div 9 &= 5 \end{aligned}$$



Care must be taken to see that children understand that the distribution can be made only on the dividend and the renaming of the dividend must make the division of each part exact.

2. ALGORITHMS FOR DIVISION

Algorithms with factors less than base: When children understand the meaning of division, its inverse, multiplication, may be capitalized upon. Familiarity with the chart of multiplication facts will point up this relationship and demonstrate that division facts are learned at the same time as the multiplication facts. The more children use what they know about multiplication the less time they will need to spend on division facts.

Children need many experiences telling simple word or story problems involving situations requiring division; writing mathematical sentences for these problems; solving the problems; and testing solutions for reality.

Before considering division with non-zero remainders, children need many experiences finding the greatest whole number that makes a sentence like the following true:

$\square \times 3 < 16$ $\square \times 5 < 26$

$\square \times 8 < 51$ $\square \times 9 < 82$

When understanding of this type of estimation has been developed, examples with non-zero remainders may be introduced. For example:

$$\begin{array}{r|l} 3\overline{)16} & \\ 15 & 5 \\ \hline 1 & \end{array}$$

Children can be led to see how the estimation using the largest whole number helps to find the quotient. As the work is recorded, attention should be called to the remainder. When the remainder is less than the divisor, the operation is complete.

Algorithms with known factor equal to base: By examining number sentences like the following, children can see a pattern develop:

$10 \div 10 = 1$ $90 \div 10 = 9$

$20 \div 10 = 2$ $150 \div 10 = 15$

$30 \div 10 = 3$ $230 \div 10 = 23$

If number sentences like the following are next introduced, children can see how a pattern involving remainders develops:

$10 \div 10 = 1$ because $10 = 1 \times 10$

$11 \div 10 = 1$ r 1 because $11 = (1 \times 10) + 1$

$12 \div 10 = 1$ r 2 because $12 = (1 \times 10) + 2$

$26 \div 10 = 2$ r 6 because $26 = (2 \times 10) + 6$

$153 \div 10 = 15$ r 3 because $153 = (15 \times 10) + 3$

$928 \div 10 = 92$ r 8 because $928 = (92 \times 10) + 8$

Children may develop a table to show the remainders when using various divisors:

Divisor	Possible Remainders	Number of Possible Remainders
2	0, 1	2
3	0, 1, 2	3
4	0, 1, 2, 3	4
5	0, 1, 2, 3, 4	5
.	.	.
.	.	.
.	.	.

Algorithms with factors greater than base: Many children find division using larger numbers difficult. To insure success it is important to develop carefully and sequentially the ideas involved. Children need many experiences answering such questions as the following:

a. What is the greatest whole number that makes the sentence $n \times 7 < 47$ true?

b. What is the greatest multiple of 10 that makes the sentence $n \times 7 < 216$ true?

c. What is the greatest multiple of 100 that makes the sentence $n \times 3 < 602$ true?

Children need help in finding the number of places a quotient will have. For example, $234 \div 7$ will have a two-place quotient because $10 \times 7 < 234$ and

100 × 7 > 234. Therefore, the answer will be a two-place numeral, more than 10, but less than 100. There should be many experiences of this type to help children determine the number of places in the quotient. As these experiences are examined and discussed, patterns will emerge:

—)— quotient will have 1 place

—)— — quotient will have 1 or 2 places*

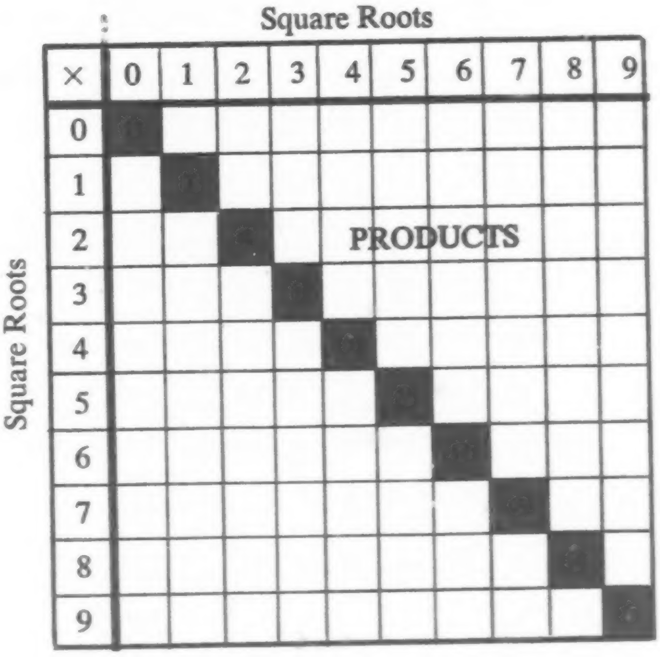
—)— — — quotient will have 2 or 3 places

*Use the multiplication chart to see where the pattern changes for each one-place divisor.

Although children will be expected ultimately to use the conventional algorithm for division, it is far better that they continue to use a longer, less efficient form with understanding than to use the conventional algorithm mechanically.

3. Square Root of a Number

The multiplication chart shows the square root of certain products. For example, since 49 is the product of two like factors, 7, the square root of 49 is 7.



Square Roots

×	0	1	2	3	4	5	6	7	8	9
0	[illegible]									
1		[illegible]								
2			[illegible]			PRODUCTS				
3				[illegible]						
4					[illegible]					
5						[illegible]				
6							[illegible]			
7								[illegible]		
8									[illegible]	
9										[illegible]

Square Roots

RATIONAL NUMBERS

Many operations can be performed by using whole numbers. As work with problems increases and mathematical experience broadens, it becomes evident that there is need for other sets of numbers. For example, there are measuring situations for which the whole numbers are insufficient. In subtraction there are pairs of whole numbers for which there is no answer in the set of whole numbers, $3 - 5 = \square$. Similarly, in division, there are pairs of whole numbers which have no quotient in the set of whole numbers, $5 \div 2 = \square$ or $9 \div 13 = \square$. These and other situations indicate that additional sets of numbers are needed in order to complete problems like those above.

The set of numbers conventionally introduced after the set of whole numbers is the set of positive rational numbers and zero, which are the fractional numbers. Properties for operations on fractional numbers are developed from properties for operations on whole numbers.

These two sets of numbers, the whole numbers and the fractional numbers, are studied in detail in the elementary school and form the major portion of elementary mathematics. Included in the study of fractional numbers is decimal notation. A third set of numbers, the integers, is introduced in the elementary school. Integers and the real numbers are studied in detail in the junior and senior high school.

Each succeeding number system represents an extension of the system previously studied in that definitions and properties of earlier systems are used to develop subsequent systems. Careful development of the systems of whole numbers and rational numbers, particularly fractional numbers, is necessary in a program of elementary mathematics.

Mathematical Ideas | **Illustrations and Explanations**

Fractional Numbers

1. Nature of Fractional Numbers

An *ordered pair* of elements is a pair of elements with one designated as first and the other designated as second.

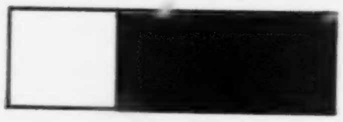
(Dick, Joe) shows Dick first and Joe second.

(cart, car) shows cart first and car second.

(2, 3) shows 2 first and 3 second.

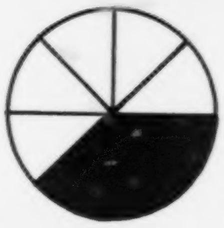
Ordered pairs of numbers with the first number a whole number and the second number a counting number constitute the beginning idea of fractional number. Ordered pairs of numbers can be related to regions, segments, and sets of objects.

Regions



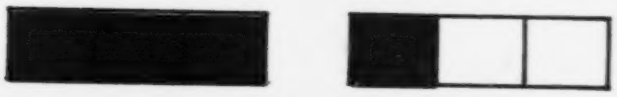
3 parts of the same size with 2 shaded

(2, 3) shows relationship between shaded parts and unit



8 parts of the same size with 3 shaded

(3, 8) shows relationship between shaded parts and unit



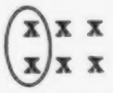
(4, 3) shows each unit separated into 3 parts of the same size with 4 of these parts considered.

Segments

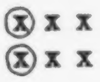
(1, 2) shows 1 of 2 parts of the same length.

The ordered pair (a, b) as applied to a unit region or a unit segment shows that the unit is split into *b* parts with *a* of them being considered.

Sets of Objects



(1, 3) shows 1 of 3 equivalent sets



(2, 6) shows 2 of 6 equivalent sets

A fraction is a symbol that names an ordered pair of numbers. It also names the fractional number.

Ordered Pair	**Fraction**	
(2, 3)	$\frac{2}{3}$	Read two thirds or 2 divided by 3
(3, 8)	$\frac{3}{8}$	Read three eighths or 3 divided by 8
(4,3)	$\frac{4}{3}$	Read four thirds or 4 divided by 3

The bar separating the two whole numbers is a symbol of division as in the division sign,"÷."

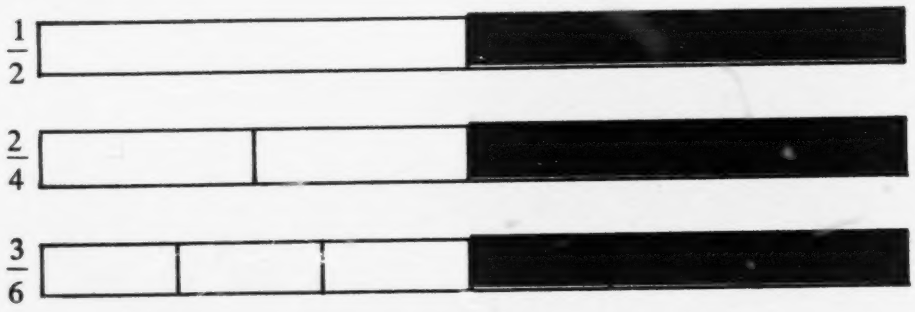
The quotient of any two whole numbers may be expressed as a fractional number.

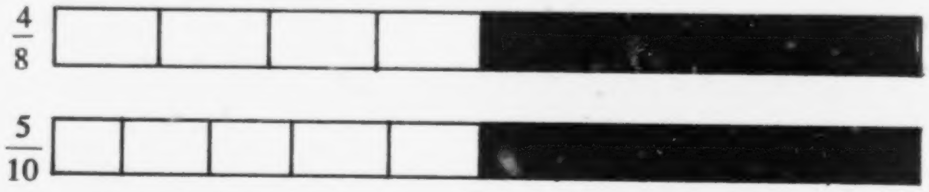
$2 \div 3 = \frac{2}{3}; 4 \div 3 = \frac{4}{3}; 8 \div 4 = \frac{8}{4}$

In the ordered pair (2, 3) the first number, 2, is called the numerator and the second number, 3, is called the denominator.

The ordered pair (a, b) is shown as the fraction $\frac{a}{b}$ where *a, b* are whole numbers, $b \neq 0$. The first member of the ordered pair (a) is called the numerator and the second number (b) is called the denominator.

A fractional number can be thought of as representing a *part,* portion or amount of the region, segment, or set. An infinite number of ordered pairs names the same fractional number.





Each shaded bar is the same part or amount of the unit region but each part is named with a different fraction. Each fraction names the same **fractional number**.

Equivalent fractions: Fractions that name the same fractional number are called equivalent fractions.

The fractions associated with each region above are different names for the same fractional number: $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10}$. The fractions are equivalent.

An equivalent fraction for any fractional number can be obtained by:

- Multiplying the numerator and denominator by the same counting number using the properties of one, that one times a number is that number and that a number divided by itself is one.
- Dividing the numerator and denominator by the same counting number.

Example:

Fractional numbers equivalent to $\frac{1}{2}$:

$$\frac{1}{2} \times 1 \text{ or } \frac{1 \times 2}{2 \times 2} = \frac{2}{4} \qquad \frac{2}{2} = 1$$

$$\frac{1 \times 5}{2 \times 5} = \frac{5}{10} \qquad \frac{5}{5} = 1$$

$$\frac{5}{15} \div 1 \text{ or } \frac{5 \div 5}{15 \div 5} = \frac{1}{3} \qquad \frac{5}{5} = 1$$

Two fractions with like denominators are equivalent if and only if their numerators are the same.

Example: $\frac{3}{4} = \frac{3}{4}$ Since denominators are alike and numerators are the same, $\frac{3}{4} = \frac{3}{4}$.

$$\frac{5}{8} = \frac{10}{16} \qquad \frac{5}{8} = \frac{10}{16}$$

$$\frac{5 \times 2}{8 \times 2} = \frac{10 \times 1}{16 \times 1} \qquad \frac{10}{16} = \frac{10 \div 2}{16 \div 2} = \frac{5}{8}$$

$$\frac{10}{16} = \frac{10}{16} \qquad \frac{5}{8} = \frac{5}{8}$$

$$\frac{5}{8} = \frac{15}{24}$$

$$\frac{5 \times 24}{8 \times 24} = \frac{15 \times 8}{24 \times 8}$$

$$\frac{5 \times 24}{192} = \frac{15 \times 8}{192}$$

$$\frac{120}{192} = \frac{120}{192}$$

$$\frac{a}{b} = \frac{c}{d}$$

$$\frac{a \times d}{b \times d} = \frac{c \times b}{d \times b}$$

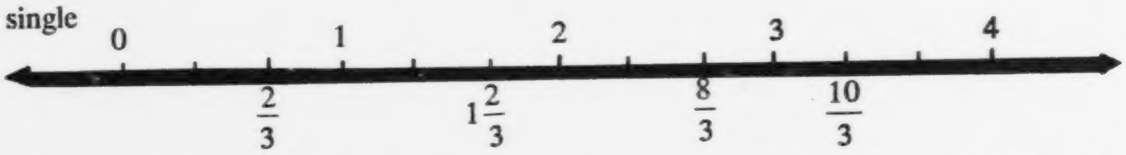
Any fractional numbers $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent if and only if $a \times d = b \times c$

Any one unknown term of two equivalent fractions may be found by using the idea that two fractional numbers are equivalent if and only if $a \times d = b \times c$.

Examples:

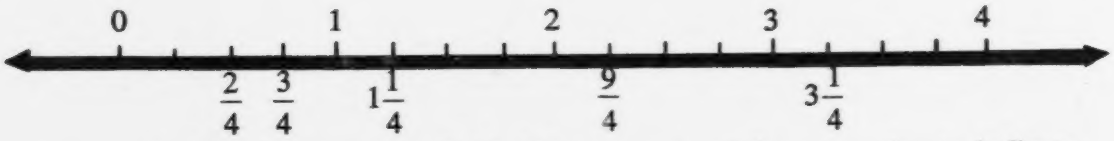
$\frac{4}{7} = \frac{n}{35}$	$4 \times 35 = 7 \times n$	$a = 4$ $b = 7$	$c = n$ $d = 35$
$\frac{4}{7} = \frac{20}{n}$	$4 \times n = 7 \times 20$		
$\frac{n}{7} = \frac{20}{35}$	$n \times 35 = 7 \times 20$		
$\frac{4}{n} = \frac{20}{35}$	$4 \times 35 = n \times 20$		

Along the number line: Each whole number can be matched with a single point.



Each fractional number can be matched with a single point.

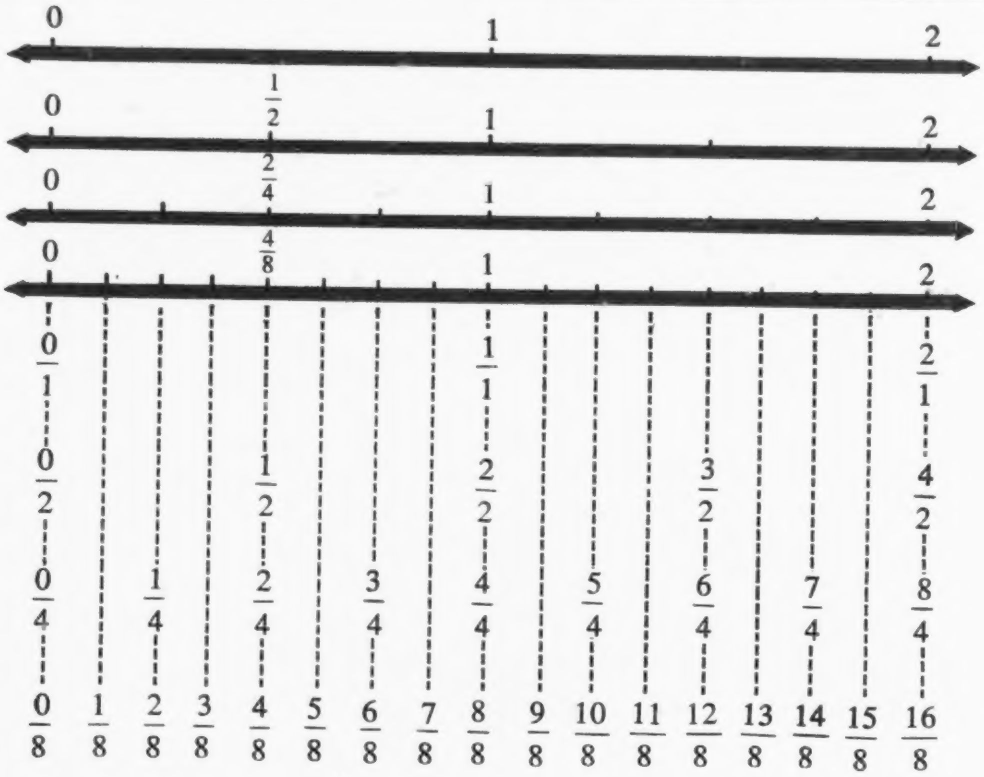
Each unit segment is divided into 3 parts of the same size to indicate thirds.



Each unit segment is divided into 4 parts of the same size to indicate fourths.

The fraction $\frac{a}{b}$ is matched with a point on the number line by splitting the unit segment into *b* pieces of the same size and matching the point at the end of *a* pieces.

Each fractional number can be named by an infinite number of ordered pairs of numbers.


0
1
2
$\frac{1}{2}$
$\frac{2}{4}$
$\frac{4}{8}$
$\frac{0}{1}$ $\frac{1}{1}$ $\frac{2}{1}$
$\frac{0}{2}$ $\frac{1}{2}$ $\frac{2}{2}$ $\frac{3}{2}$ $\frac{4}{2}$
$\frac{0}{4}$ $\frac{1}{4}$ $\frac{2}{4}$ $\frac{3}{4}$ $\frac{4}{4}$ $\frac{5}{4}$ $\frac{6}{4}$ $\frac{7}{4}$ $\frac{8}{4}$
$\frac{0}{8}$ $\frac{1}{8}$ $\frac{2}{8}$ $\frac{3}{8}$ $\frac{4}{8}$ $\frac{5}{8}$ $\frac{6}{8}$ $\frac{7}{8}$ $\frac{8}{8}$ $\frac{9}{8}$ $\frac{10}{8}$ $\frac{11}{8}$ $\frac{12}{8}$ $\frac{13}{8}$ $\frac{14}{8}$ $\frac{15}{8}$ $\frac{16}{8}$

A fraction is said to be in its simplest form when the numerator and denominator are *relatively prime*—have no common factor greater than one.

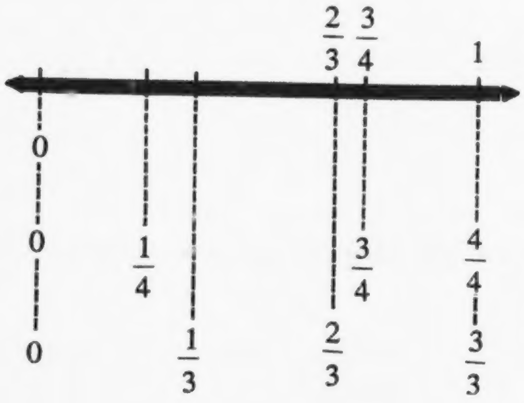
$\frac{8}{12} = \frac{4 \times 2}{4 \times 3}$ 4 is a common factor, therefore

$\frac{8}{12} = \frac{8 \div 4}{12 \div 4} = \frac{2}{3}$ (the simplest form)

$\frac{9}{10} = \frac{3 \times 3}{5 \times 2}$ (no common factor— $\frac{9}{10}$ is the simplest form.)

9 and 10 are relatively prime.

Order of fractional numbers: The order of fractional numbers can be shown using the number line. A given number is less than each number to its right and greater than each number to its left.


$\frac{2}{3}$ $\frac{3}{4}$
1
0
0
0
$\frac{1}{4}$
$\frac{1}{3}$
$\frac{2}{3}$
$\frac{3}{4}$
$\frac{4}{4}$
$\frac{3}{3}$

Since the length of the segment associated with $\frac{3}{4}$ is longer than the segment associated with $\frac{2}{3}$, $\frac{3}{4} > \frac{2}{3}$ and $\frac{2}{3} < \frac{3}{4}$.

Two fractional numbers can be compared by renaming one or both so that they have the same denominators. Comparing the numerators will show which is greater.

Example: Compare $\frac{2}{3}$ and $\frac{3}{4}$

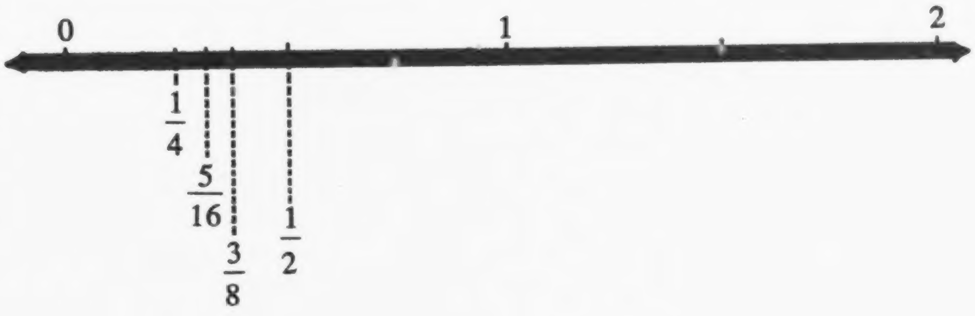
$\frac{2}{3} = \frac{8}{12}$ $\frac{3}{4} = \frac{9}{12}$

Since $9 > 8$, $\frac{3}{4} > \frac{2}{3}$

Given any two fractions $\frac{a}{b}$ and $\frac{c}{d}$ one of the following is true:

$$\frac{a}{b} = \frac{c}{d},\ \frac{a}{b} > \frac{c}{d},\ \text{or}\ \frac{a}{b} < \frac{c}{d}$$

Density of fractional numbers: Another fractional number can always be found between two given fractional numbers. This property is known as density. Whole numbers are not dense because another whole number cannot be found between two consecutive whole numbers.

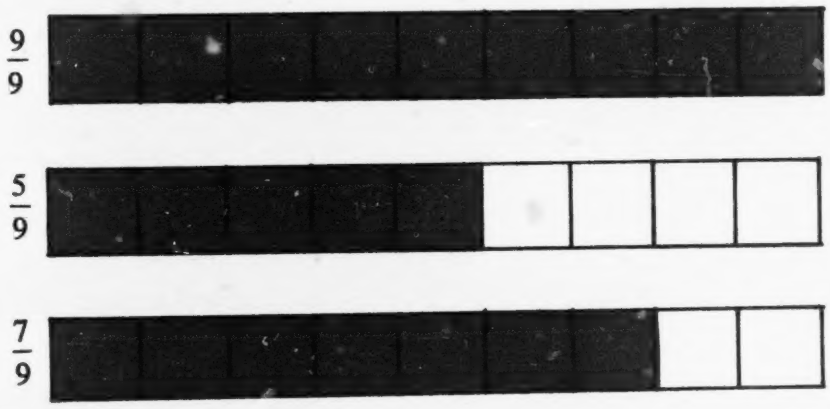


Between $\frac{1}{4}$ and $\frac{1}{2}$ is the fractional number $\frac{3}{8}$. Between $\frac{1}{4}$ and $\frac{3}{8}$ is the fractional number $\frac{5}{16}$. This process can be continued indefinitely, leading to the idea that there is an infinite number of fractional numbers between any two given fractional numbers.

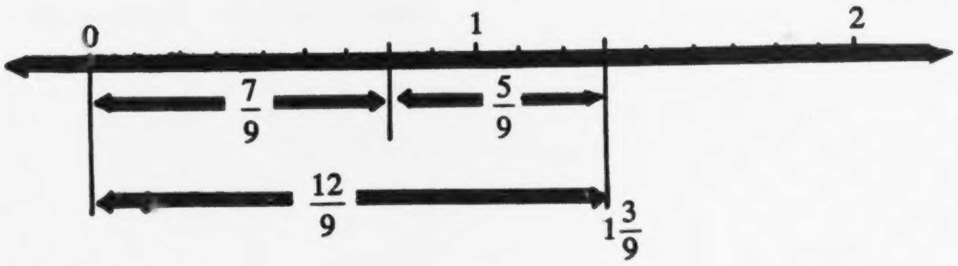
2. **Operations on Fractional Numbers**

Addition: The addition of fractional numbers, as with the addition of whole numbers, is the binary operation on two numbers called addends to produce a third number called the sum.

To add fractional numbers named by fractions with the same denominators, the numerators are added and the same denominator used.



Therefore $\frac{7}{9} + \frac{5}{9} = \frac{12}{9}$



Given two fractions $\frac{a}{b}$ and $\frac{c}{b}$ with the same non-zero denominator, $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$.

To add fractional numbers with unlike denominators the addends are renamed so that all denominators are alike, and numerators are added as with fractions having like denominators.

Examples:

$\frac{3}{4} + \frac{7}{8} = \square$ $\qquad$ $\frac{3}{4} = \frac{3 \times 2}{4 \times 2} = \frac{6}{8}$

$\frac{6}{8} + \frac{7}{8} = \frac{13}{8}$

$\frac{5}{6} + \frac{1}{7} = \square$ $\qquad$ $\frac{5}{6} = \frac{5 \times 7}{6 \times 7} = \frac{35}{42}$

$\frac{1}{7} = \frac{1 \times 6}{7 \times 6} = \frac{6}{42}$

$\frac{35}{42} + \frac{6}{42} = \frac{41}{42}$

Another way to rename addends so that all denominators are alike, uses the idea of the least common multiple. (l.c.m.)

Examples:

$\frac{1}{6} + \frac{4}{9} = \square$ $\qquad$ To find the l.c.m. of 6 and 9:

Counting number multiples of 6 = 6, 12, 18, 24, 30, 36, 42 . . .

Counting number multiples of 9 = 9, 18, 27, 36, 45, 54 . . .

l.c.m. of 6 and 9 = 18

Hence the least common multiple of the denominators is 18.

$\frac{1}{6} + \frac{4}{9} = \frac{3}{18} + \frac{8}{18} = \frac{11}{18}$

$\frac{1}{6} + \frac{4}{9} + \frac{5}{12} =$ $\qquad$ To find the l.c.m. of 6, 9, and 12:

Counting number multiples of 6 = 6, 12, 18, 24, 30, 36 . . .

Counting number multiples of 9 = 9, 18, 27, 36, 45, 54 . . .

Counting number multiples of 12 = 12, 24, 36, 48, 60 . . .

l.c.m. of 6, 9, and 12 = 36

$$\frac{1}{6} + \frac{4}{9} + \frac{5}{12} = \frac{6}{36} + \frac{16}{36} + \frac{15}{36} = \frac{37}{36}$$

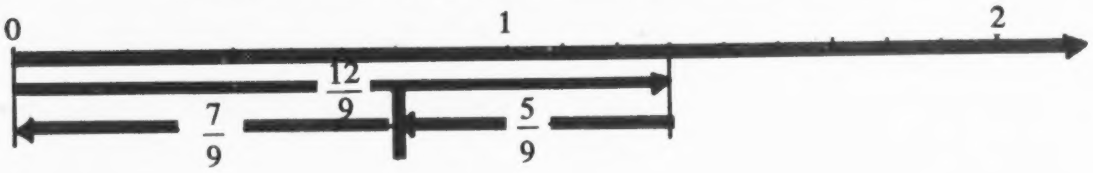
Subtraction: The subtraction of fractional numbers, as with the subtraction of whole numbers, is the inverse operation of addition and is the operation of finding an addend when the sum and the other addend are known.

To subtract fractional numbers named by fractions with the same denominator, the numerators are subtracted and the like denominator used.

$\frac{12}{9} - \frac{5}{9} = \frac{7}{9}$ Since subtraction is the inverse of addition

$\frac{7}{9} + \frac{5}{9} = \frac{12}{9}$

A picture of the separation of parts in subtraction can be shown by movement to the left on the number line.



$$\frac{12}{9} - \frac{5}{9} = \frac{7}{9}$$

Given *a, b, c,* whole numbers $b \neq 0$ and $a > c$:

$$\frac{a}{b} - \frac{c}{b} = \frac{a - c}{b}$$

$a > c$ assures that $a - c$ is an element of the set of whole numbers, so that the result of subtracting fractional numbers, under the stated restrictions, is another fractional number.

To subtract fractional numbers named by fractions with unlike denominators, the sum and the known addend are renamed so that denominators are alike in the same manner as described for addition. Then the subtraction is performed as with fractions having like denominators.

Multiplication: Multiplication of fractional numbers, as on the set of whole numbers, is the binary operation on two numbers called factors to produce a third number called the product.

To multiply fractional numbers, the numerators are multiplied and then the denominators are multiplied.

$$\frac{1}{2} \times \frac{2}{3} = \frac{1 \times 2}{2 \times 3} = \frac{2}{6}$$

An algorithm for multiplication of fractional numbers can be developed by capitalizing on the student's knowledge of the use of "of" in a mathematical sentence such as:

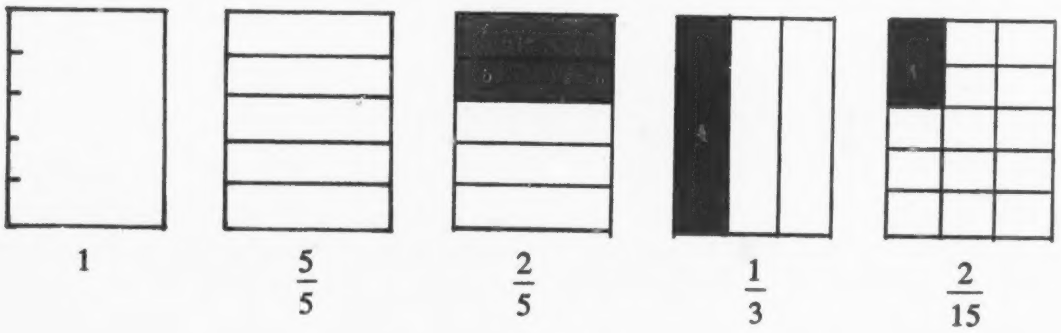
$$\frac{1}{2} \text{ of } 4 = 2; \qquad \frac{1}{2} \text{ of } 10¢ = 5¢$$

The use of "of" in a mathematical sentence may be interpreted through a pattern as follows:

$\frac{1}{2}$ of $8 = 4$	$8 \times 6 = 48$ $4 \times 6 = 24$	$\frac{1}{2}$ of $48 = 24$
$\frac{1}{2}$ of $4 = 2$	$2 \times 6 = 12$	$\frac{1}{2}$ of $24 = 12$
$\frac{1}{2}$ of $2 = 1$	$1 \times 6 = 6$	$\frac{1}{2}$ of $12 = 6$
$\frac{1}{2}$ of $1 = \frac{1}{2}$	$\frac{1}{2} \times 6 = 3$	$\frac{1}{2}$ of $6 = 3$

Where one factor is halved, the product is halved. As the pattern continues, it can be seen that $\frac{1}{2} \times 6$ means the same as $\frac{1}{2}$ of 6.

To find $\frac{1}{3}$ of $\frac{2}{5}$ use the model:

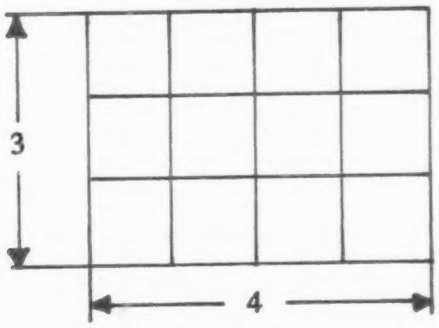


Hence, $\frac{1}{3} \times \frac{2}{5} = \frac{2}{15}$

It can be observed that the numerator of the product equals 1×2 and the denominator equals 3×5, thus verifying that $\frac{1}{3} \times \frac{2}{5} = \frac{1 \times 2}{3 \times 5} = \frac{2}{15}$

Another way to illustrate the algorithm for the multiplication of fractional numbers is to use a rectangular region.

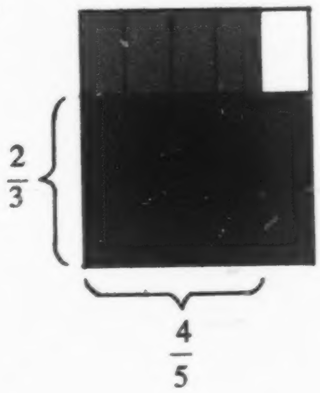
The area of a rectangular region with side measures that are whole numbers is found by multiplication.



Area = 3 rows of 4 sq. units each

Area = 3 × 4 = 12 sq. units

It is reasonable to use multiplication to find the area if the side measures are fractional numbers.



Area of the region is 1 square unit. Side measures are 1 linear unit. The shaded region measures $\frac{2}{3}$ by $\frac{4}{5}$. The area of the rectangular region measuring $\frac{2}{3}$ by $\frac{4}{5}$ is $\frac{8}{15}$ of the square unit. Hence, it is sensible for $\frac{2}{3} \times \frac{4}{5}$ to be $\frac{8}{15}$.

The product of the denominators shows the number of equal-sized pieces in the unit square. The product of the numerators shows the number of pieces in the smaller region.

$$\frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5} = \frac{8}{15}$$

If $\frac{a}{b}$ and $\frac{c}{d}$ are fractional numbers, the product is $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$

Division: The division of fractional numbers, as in the division of whole numbers, is the inverse operation of multiplication and is the operation of finding a factor when the product and one factor are known.

Division of fractional numbers may be approached in more than one way. Mathematical ideas used in these approaches are:

1. The quotient of any two numbers may be expressed as a fraction:

$$a \div b = \frac{a}{b},\ b \neq 0$$

2. Any number divided by itself has 1 as a quotient:

$$\frac{a}{a} = 1, a \neq 0$$

3. Any number multiplied by 1 has itself as product:

$$\frac{a}{b} \times 1 = \frac{a}{b}$$

4. Any number divided by 1 has itself as quotient:

$$\frac{a}{b} \div 1 = \frac{a}{b}$$

5. The *reciprocal* of a number is that number by which the first number is multiplied to give the product, 1:

$$\frac{a}{b} \times \frac{b}{a} = \frac{a \times b}{b \times a} = \frac{a \times b}{a \times b} = 1, a \neq 0, b \neq 0$$

6. The product of two fractional numbers is the product of the numerators divided by the product of the denominators:

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

7. The quotient of two fractional numbers is the quotient of the numerators divided by the quotient of the denominators:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d}$$

The quotient of two fractional numbers may be expressed as another fractional number, frequently called a complex fraction:

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}}$$

8. The inverse operation of division is multiplication:

$$\frac{a}{b} = c \text{ and } c \times b = a$$

The division algorithm may be developed by the use of a fraction where the terms are fractional numbers, often called a complex fraction:

$$\frac{1}{2} \div \frac{1}{4} = \frac{\frac{1}{2}}{\frac{1}{4}}$$

(1) Numbers refer to ideas on pages 61-62.

$$= \frac{\frac{1}{2}}{\frac{1}{4}} \times \frac{\frac{4}{1}}{\frac{4}{1}} \qquad (3)\ (2)$$

$$= \frac{\frac{1}{2} \times \frac{4}{1}}{\frac{1}{4} \times \frac{4}{1}} \qquad (6)$$

$$= \frac{\frac{1}{2} \times \frac{4}{1}}{1} \qquad (5)$$

$$= \frac{1}{2} \times \frac{4}{1} \qquad (6)$$

$$= \frac{4}{2} = 2$$

Division by a fractional number is the same as multiplication by its reciprocal.

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} \qquad (1)$$

$$= \frac{\frac{a}{b}}{\frac{c}{d}} \times \frac{\frac{d}{c}}{\frac{d}{c}} \qquad (3)\ (2)$$

$$= \frac{\frac{a}{b} \times \frac{d}{c}}{\frac{c}{d} \times \frac{d}{c}} \qquad (6)$$

$$= \frac{\frac{a}{b} \times \frac{d}{c}}{1} \qquad (5)$$

$$= \frac{a}{b} \times \frac{d}{c} \qquad (6)$$

Division by $\frac{c}{d}$ is the same as multiplication by its reciprocal, $\frac{d}{c}$.

Another approach uses the common denominator method. This method involves renaming the two fractions so that denominators are alike, then dividing the whole number numerators and denominators.

$\frac{1}{2} \div \frac{1}{4} = \square$ $\qquad \frac{1}{2} \times \frac{2}{2} = \frac{2}{4}$ (2) (3)

$\frac{2}{4} \div \frac{1}{4} = \frac{2 \div 1}{4 \div 4}$ (7)

$= \frac{2 \div 1}{1}$

(4)

$= 2$

$\frac{a}{b} \div \frac{c}{b} = \frac{a \div c}{b \div b}$ (7)

$= \frac{a \div c}{1}$ (4)

Inverse operation approach may also be used in developing the division algorithm.

$\frac{2}{3} \div \frac{5}{7} = \square$ means $\square \times \frac{5}{7} = \frac{2}{3}$

In order to make this mathematical sentence true, $\square$ must be a number which has $\frac{2}{3}$ as a factor.

$\square$ must be $(\frac{2}{3} \times \triangle)$

The mathematical sentence becomes $(\frac{2}{3} \times \triangle) \times \frac{5}{7} = \frac{2}{3}$

Applying the associative property:

$$\frac{2}{3} \times (\triangle \times \frac{5}{7}) = \frac{2}{3}$$

If $\frac{2}{3} \times (\quad) = \frac{2}{3}$, then $(\quad) = 1$ and $(\triangle \times \frac{5}{7}) = 1$

Therefore, $\triangle = \frac{7}{5}$ since $\frac{7}{5} \times \frac{5}{7} = 1$

$$(\frac{2}{3} \times \frac{7}{5}) \times \frac{5}{7} = \frac{2}{3}$$

Then $(\frac{2}{3} \times \frac{7}{5}) = \square$ $\qquad$ Therefore, $\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \times \frac{7}{5}$

PROPERTIES OF ADDITION AND SUBTRACTION OF FRACTIONAL NUMBERS

The same properties that hold true for addition and subtraction with respect to the set of whole numbers also hold true for addition and subtraction with respect to the set of fractional numbers. In brief these are:

Addition	*Subtraction*
1. Closure	1. Not closed
2. Commutativity	2. Not commutative
3. Associativity	3. Not associative
4. Identity element (zero)	4. No identity ($a - 0 = a$, but $0 - a \neq a$)
5. No inverse element	5. No inverse element

PROPERTIES OF MULTIPLICATION AND DIVISION OF FRACTIONAL NUMBERS

The properties that hold true for multiplication and division with respect to the set of whole numbers also hold true for multiplication and division with respect to the set of fractional numbers.

Multiplication	*Division*
1. Closure	1. Closure
2. Commutativity	2. Not commutative
3. Associativity	3. Not associative
4. Identity element (one)	4. Identity element (one) on the right only $\frac{a}{b} \div 1 = \frac{a}{b}$, but $1 \div \frac{a}{b} \neq \frac{a}{b}$
5. Distributivity over addition	5. Distributivity over addition from the right hand side only
6. Distributivity over subtraction	6. Distributivity over subtraction from the right hand side only
7. Multiplication property of zero	
8. Inverse	

The reciprocal or inverse property is true under the multiplication of fractional numbers but not for whole numbers. It states: If *a* and *b* are whole numbers, *a* and $b \neq 0$,

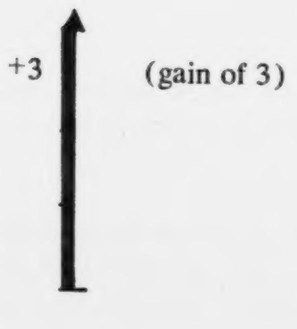
$$a \times \frac{1}{a} = 1 \text{ and } \frac{a}{b} \times \frac{b}{a} = 1$$

$\frac{1}{a}$ is called the reciprocal of *a* and *a* is called the reciprocal of $\frac{1}{a}$.

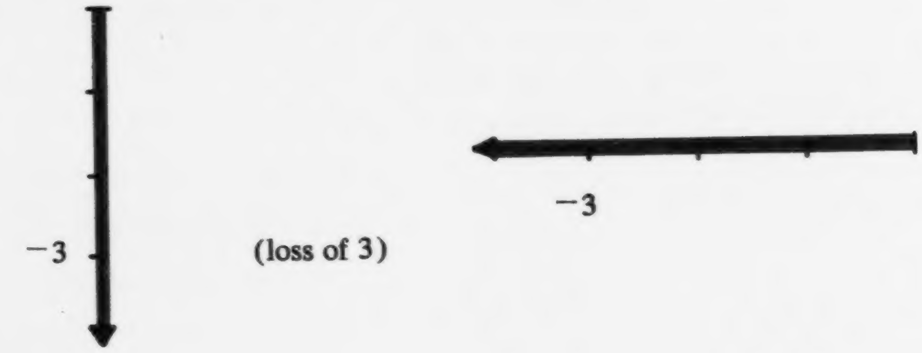
$\frac{b}{a}$ and $\frac{a}{b}$ are reciprocals of each other.

The reciprocal of a number is the number by which the first number must be multiplied to give a product of 1—the identity element. Every fractional number, except 0, has a reciprocal. Another name for the reciprocal property is the *multiplicative inverse* property.

The closure property is added to the list of properties under division for fractional numbers. Since the division of one fractional number by another fractional number, excluding zero as a divisor, will always yield a fractional number as the quotient, the set of fractional numbers is said to be closed with respect to division.

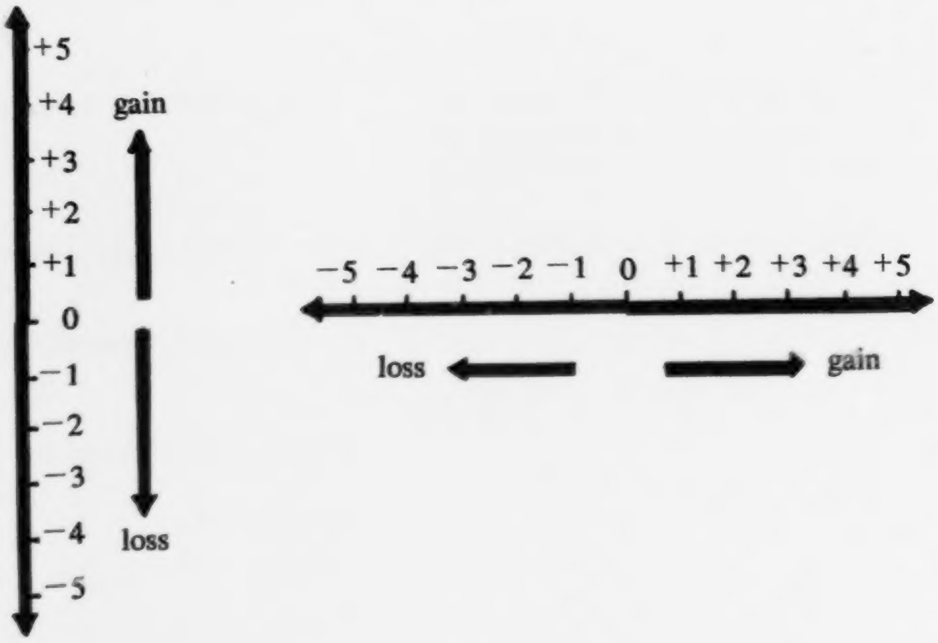
Mathematical Ideas	Illustrations and Explanations
The Integers	
1. NATURE OF INTEGERS	
The set of integers consists of the set of whole numbers and the set of negatives of the natural or counting numbers.	. . . $^{-}5$, $^{-}4$, $^{-}3$, $^{-}2$, $^{-}1$, 0, 1, 2, 3, 4, 5, . . .
The positive integers are the same as the set of counting numbers. Often for emphasis the raised plus sign is used for positive numbers. The number $^{+}5$ is read "positive five."	1, 2, 3, 4, 5, . . . or $^{+}1$, $^{+}2$, $^{+}3$, $^{+}4$, $^{+}5$, . . .
The raised dash denotes a negative number. The integer $^{-}5$ is read "negative five."	. . . $^{-}5$, $^{-}4$, $^{-}3$, $^{-}2$, $^{-}1$
It should be kept in mind that these raised symbols, $^{+}$ and $^{-}$, are part of the names of the numbers and not operational symbols. It is agreed that, if a numeral has no sign, it is a positive integer.	
Positive and negative integers may be used to denote change in direction from one position to another.	
Positive may indicate a change that is an increase.	Growth of 2 inches; increasing change of 2; $^{+}2$ Deposit of \$50; increasing change of \$50; $^{+}50$ 95 feet above sea level; $^{+}95$
Negative may indicate a change that is a decrease.	Loss of \$75; decreasing change of \$75; $^{-}75$ Temperature of 5° below zero; decreasing change of 5° from 0; $^{-}5$ 15 in the hole; decreasing change of 15 from 0; $^{-}15$
Arrows may be used to represent change. Arrows pointing to the right or up indicate positive integers.	

Arrows pointing to the left or down indicate negative integers.



(loss of 3)

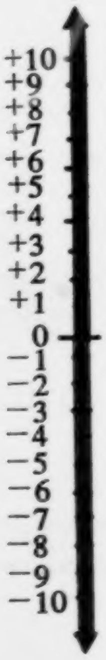
Points on the number line that match positive and negative integers indicate change from 0.



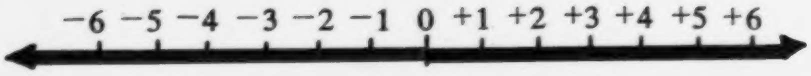
+5
+4
+3
+2
+1
0
−1
−2
−3
−4
−5
gain
loss
−5 −4 −3 −2 −1 0 +1 +2 +3 +4 +5
loss
gain

2. ORDER OF INTEGERS

On the number line any integer is less than any other integer to the right and is greater than any to its left. Any integer is less than any integer above it and greater than any below it on the vertical number line.



+10
+9
+8
+7
+6
+5
+4
+3
+2
+1
0
−1
−2
−3
−4
−5
−6
−7
−8
−9
−10



−6 −5 −4 −3 −2 −1 0 +1 +2 +3 +4 +5 +6

Loss of 3 is less than a loss of 1. $^{-}3 < {}^{-}1$.

A temperature of $^{+}3°$ is higher than a temperature of $^{-}7°$. $^{+}3 > {}^{-}7$

$0 > {}^{-}2$ $^{-}2 < 0$

$^{+}15 > {}^{-}17$ $^{-}1 < 0$

Mathematical Ideas | **Illustrations and Explanations**

Two integers the same distance from 0, one to the right, the other to the left on the number line, are called opposites or inverses.

On number lines above the following opposites may be noted:

Opposites

$^{-}3$ and $^{+}3$ are opposites

$^{-}5$ and $^{+}5$ are opposites

$^{-}9$ and $^{+}9$ are opposites

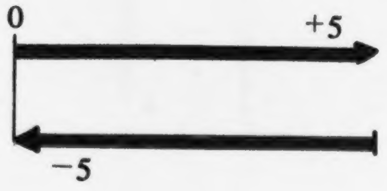
$^{-}10$ and $^{+}10$ are opposites

0 is neither positive nor negative; it is its own opposite.

The Additive Inverse Property: The additive inverse of an integer is the same as the opposite of the integer.

The sum of a number and its additive inverse is 0.

$^{-}6$ is the additive inverse of $^{+}6$



$^{+}5$ Gain of 5

$^{-}5$ Loss of 5

Gain of 5 then loss of 5 is 0

$(^{+}5) + (^{-}5) = 0$

$(^{+}7) + (^{-}7) = 0$

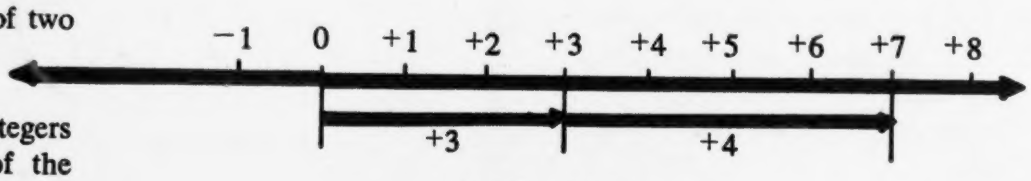
$(^{-}10) + (^{+}10) = 0$

3. OPERATIONS ON INTEGERS

Addition: The addition of integers, as with the addition of whole numbers and fractional numbers, is the operation of finding the sum of two addends.

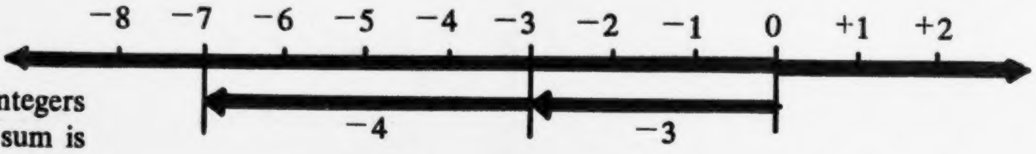
The sum of two integers may be found by using the number line.

The sum of two positive integers is the same as the sum of the equivalent counting numbers, a positive number.

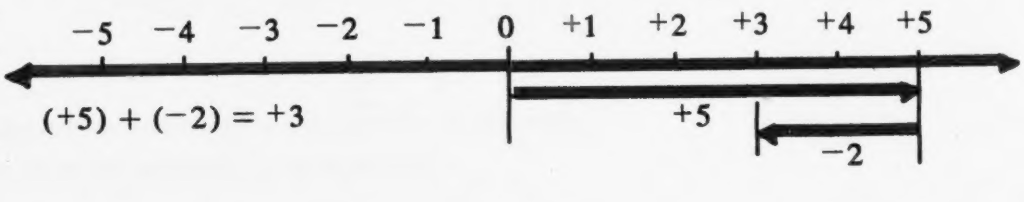
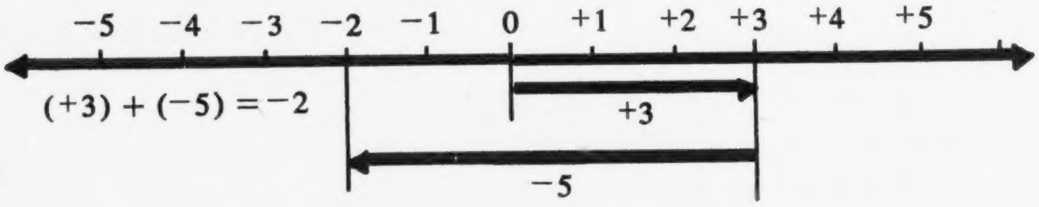
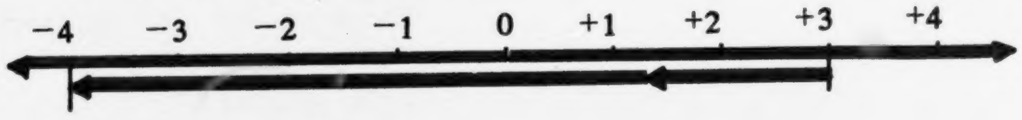


$(^{+}3) + (^{+}4) = {}^{+}7$

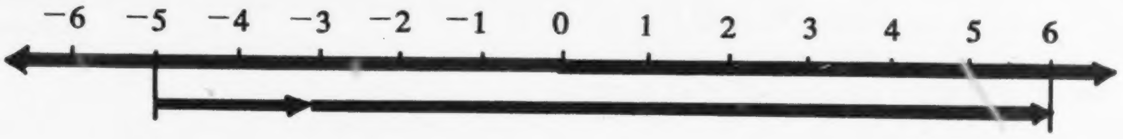
The sum of two negative integers is a negative integer. This sum is the negative of their combined distances from zero on the number line.



$(^{-}3) + (^{-}4) = {}^{-}7$

Mathematical Ideas	Illustrations and Explanations
The sum of a positive and a negative integer when the positive number is farther from 0 on the number line is a positive number. The sum is the difference of their distances from 0.	 $(+5) + (-2) = +3$
The sum of positive and negative integers when the negative integer is farther from 0 on the number line is a negative integer. The sum is the negative of the difference of the distance from 0.	 $(+3) + (-5) = -2$
The additive inverse property is useful in finding the sum of a positive and negative integer.	To find the sum $(+9) + (-3)$, $+9$ is renamed as $[(+6) + (+3)]$ $+9 + (-3) = [(+6) + (+3)] + (-3)$ $= +6 + [(+3) + (-3)]$ associative property $= +6 + 0$ additive inverse $= +6$ To find the sum $-16 + +9$, -16 is renamed as $[(-7) + (-9)]$ $(-16) + (+9) = [(-7) + (-9)] + (+9)$ $= (-7) + [(-9) + (+9)]$ associative property $= -7 + 0$ additive inverse $= -7$
Subtraction: The subtraction of integers, as with the subtraction of whole numbers and fractional numbers is the operation of finding the missing addend when the sum and one addend are known.	$-4 - (+3)$ is the number that added to $+3$ makes -4. $-4 - (+3) = \square$ means $\square + (+3) = -4$ $6 - (-5)$ is the number that added to -5 makes 6. $6 - (-5) = \square$ means $\square + (-5) = 6$ If a, b, and c are integers and $a + b = c$, then $c - b = a$
The difference $c - b$ on a number line is the change in moving from b to c. The direction of movement determines whether the missing addend is positive or negative.	$-4 - (+3) = \square$ means $\square + (+3) = -4$  From $+3$ to -4 is 7 units. The direction is negative, hence $\square = -7$ $-4 - (+3) = -7$

$6 - (^-5) = \square$ means $\square + (^-5) = 6$



From $^-5$ to $^+6$ is 11 units in a positive direction, hence $\square = {}^+11$

Using the definition of subtraction, $a + b = c$ and $c - b = a$, it can be shown that to subtract *b* from *c* the additive inverse of *b* is added to *c*.

$^-5 - (^-7) = \square$ means $\square + (^-7) = {}^-5$

$[\square + (^-7)] + (^+7) = {}^-5 + (^+7)$ (If equals are added to equals, the sums are equal. $3 + 4 + 2 = 7 + 2$)

$\square + [(^-7) + (^+7)] = {}^-5 + (^+7)$ associative property

$\square + 0 = {}^-5 + (^+7)$ identity element

$\square = {}^-5 + (^+7)$

$\square = 2$

To subtract $^-7$, add its additive inverse, $^+7$.

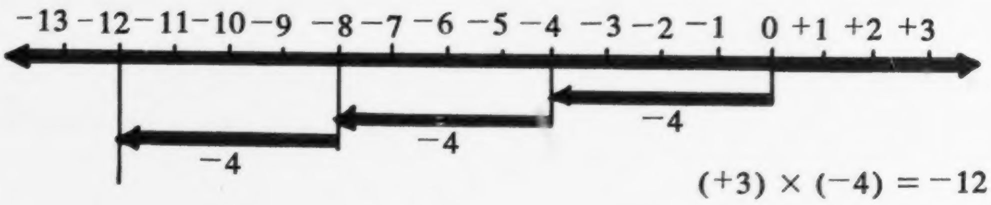
Multiplication: The multiplication of integers, as with the multiplication of whole numbers and fractional numbers, is the operation of finding the product of two factors.

The product of two positive integers is a positive integer, the same as the product of the equivalent counting numbers.

$(^+5) \times (^+6) = {}^+30$

See Operations on Whole Numbers: Multiplication

The product of a positive number and a negative number is a negative number.



$(^+3) \times (^-4) = {}^-12$

The decreasing factor approach develops a pattern which is useful in illustrating this product.

$^+2 \times {}^+3 = {}^+6$
$^+1 \times {}^+3 = {}^+3$
$0 \times {}^+3 = 0$
$^-1 \times {}^+3 = x$
$^-2 \times {}^+3 = y$

The product decreases 3 each time. If the pattern continues, x must be $^-3$ and y, $^-6$.

Mathematical Ideas	Illustrations and Explanations
The product of two negative integers is a positive number.	The decreasing factor approach is useful for finding the product of two negative numbers. $5 \times {}^-4 = {}^-20$ $4 \times {}^-4 = {}^-16$ $3 \times {}^-4 = {}^-12$ $2 \times {}^-4 = {}^-8$ $1 \times {}^-4 = {}^-4$ $0 \times {}^-4 = 0$ ${}^-1 \times {}^-4 = x$ ${}^-2 \times {}^-4 = y$ ${}^-3 \times {}^-4 = z$ Since each product increases by 4, it is sensible for x to be ${}^+4$, y to be ${}^+8$ and z be ${}^+12$.
Division: The division of integers, as with the division of whole numbers and rational numbers, is the operation of finding the missing factor when the product and one factor are known. The missing factor is called the quotient.	
The quotient of two positive integers is a positive number, as in the division of counting numbers.	${}^+15 \div {}^+3 = \square$ means $\square \times {}^+3 = {}^+15$ Since ${}^+5 \times {}^+3 = {}^+15$ ${}^+15 \div {}^+3 = {}^+5$
The quotient of a positive integer and a negative integer is a negative integer.	${}^+12 \div ({}^-4) = \square$ means $({}^-4) \times \square = {}^+12$ Since $({}^-4) \times ({}^-3) = {}^+12$ $12 \div ({}^-4) = {}^-3$
The quotient of a negative integer and a positive integer is a negative integer.	$(-12) \div ({}^+3) = \square$ means $({}^+3) \times \square = {}^-12$ Since ${}^+3 \times ({}^-4) = {}^-12$ ${}^-12 \div {}^+3 = {}^-4$
The quotient in the division of two negative integers is a positive integer.	${}^-12 \div ({}^-3) = \square$ means ${}^-3 \times \square = {}^-12$ Since $({}^-3) \times ({}^+4) = {}^-12$ $({}^-12) \div ({}^-3) = {}^+4$

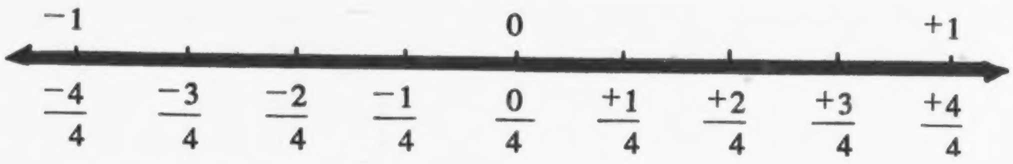
The rules for operations on positive and negative numbers should be stated only after *much* work using concrete examples and the number line as illustrated above. They may be delayed completely until junior high school with no loss. Danger exists in the formulation of generalizations too early.

Mathematical Ideas | **Illustrations and Explanations**

Negatives of the Fractional Numbers

1. THE NATURE OF NEGATIVE FRACTIONAL NUMBERS

Just as each whole number has its opposite, each fractional number has a corresponding number an equal distance to the left of 0 on the number line, called the negative of that fractional number. The sum of any pair of opposite fractional numbers is 0.



Opposites

$\frac{^-3}{4}$ and $\frac{^+3}{4}$ $\qquad$ $\frac{^-3}{4} + \frac{^+3}{4} = 0$

$\frac{^-5}{4}$ and $\frac{^+5}{4}$ $\qquad$ $\frac{^-5}{4} + \frac{^+5}{4} = 0$

2. OPERATIONS ON NEGATIVE FRACTIONAL NUMBERS

Rules which apply to operations on negative integers apply to operations on negative fractional numbers.

Addition:

Examples:

$$\frac{^-1}{4} + \frac{^-1}{3} = \frac{^-3}{12} + \frac{^-4}{12} = \frac{^-7}{12}$$

$$\frac{^+5}{8} + \frac{^-1}{4} = \frac{^+5}{8} + \frac{^-2}{8} = \frac{^+3}{8}$$

Subtraction:

Examples:

$$\frac{^-1}{4} - \frac{^-2}{3}$$

$= \frac{^-1}{4} + \frac{^+2}{3}$ additive inverse

$= \frac{^-3}{12} + \frac{^+8}{12}$ rename

$= \frac{^+5}{12}$ addition

$$\frac{^-1}{2} - \frac{^+2}{3}$$

$= \frac{^-1}{2} + \frac{^-2}{3}$ additive inverse

$= \frac{^-3}{6} + \frac{^-4}{6}$ rename

$= \frac{^-7}{6}$ addition

Multiplication:

Examples:

$$\frac{+1}{5} \times \frac{-2}{3}$$

$$= \frac{+1 \times -2}{5 \times 3}$$

$$= \frac{-2}{15}$$

$$\frac{-1}{3} \times \frac{-5}{6}$$

$$= \frac{-1 \times -5}{3 \times 6}$$

$$= \frac{+5}{18}$$

Division:

Example:

$$\frac{+5}{8} \div \frac{-1}{7}$$

$$= \frac{5}{8} \times \frac{-7}{1} \quad \text{reciprocal}$$

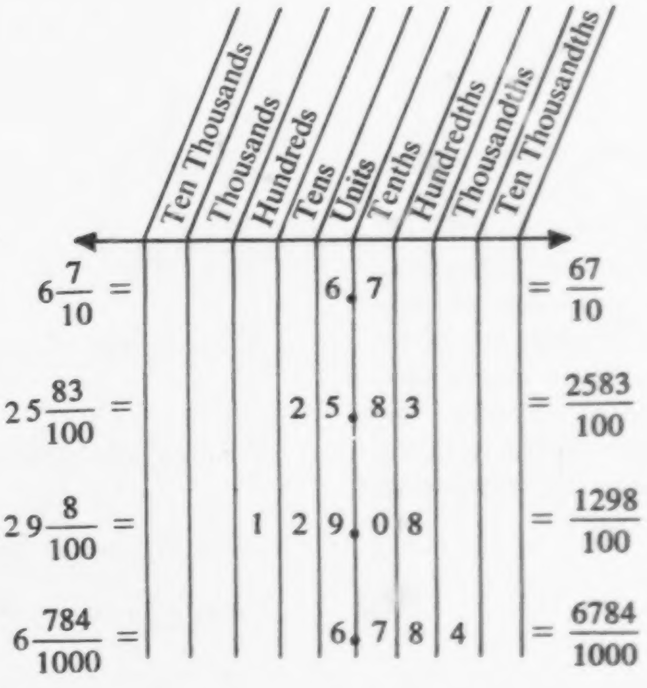
$$= \frac{5 \times -7}{8 \times 1}$$

$$= \frac{-35}{8}$$

Decimals

1. Nature of Decimals

Decimals are symbols for fractional numbers having denominators which are some power of ten. The decimal form of $\frac{3}{10}$ is .3. The denominator is not written but is indicated by the number of places to the right of the decimal point. The term decimal means that the place value notation is used to express fractional numbers.



	Ten Thousands	Thousands	Hundreds	Tens	Units	Tenths	Hundredths	Thousandths	Ten Thousandths	
$6\frac{7}{10} =$					6.	7				$= \frac{67}{10}$
$25\frac{83}{100} =$				2	5.	8	3			$= \frac{2583}{100}$
$129\frac{8}{100} =$			1	2	9.	0	8			$= \frac{1298}{100}$
$6\frac{784}{1000} =$					6.	7	8	4		$= \frac{6784}{1000}$

Mathematical Ideas	Illustrations and Explanations
Decimals can be represented as a sum of fractional numbers each of whose denominators is a power of ten.	$\frac{5}{10} = 0.5$ $\frac{75}{100} = \frac{70}{100} + \frac{5}{100} = \frac{7}{10} + \frac{5}{100} = \frac{75}{100} = 0.75$ $\frac{125}{1000} = \frac{100}{1000} + \frac{20}{1000} + \frac{5}{1000} = \frac{1}{10} + \frac{2}{100} + \frac{5}{1000} = \frac{125}{1000} = 0.125$
Every fractional number, $\frac{a}{b}$ ($b \neq 0$), can be expressed using powers of ten as denominators either as terminating decimals or as repeating decimals:	
Terminating decimals: Any rational number whose denominator has only 2, 5, or 2 and 5 as its prime factors may be written as a terminating decimal.	$\frac{1}{2} = 0.5$ $\frac{2}{5} = 0.4$ $\frac{1}{4} = \frac{1}{2 \times 2}$ $\frac{1}{4} = 0.25$ $\frac{1}{8} = \frac{1}{2 \times 2 \times 2}$ $\frac{1}{8} = 0.125$ $\frac{1}{10} = \frac{1}{2 \times 5}$ $\frac{1}{10} = 0.1$ $\frac{1}{40} = \frac{1}{2 \times 2 \times 2 \times 5}$ $\frac{1}{40} = 0.025$
Repeating decimals: Any rational number whose denominator has prime factors other than 2 or 5 may be written as a repeating decimal. In repeating decimals the sequence of digits which repeat in the quotient repeat periodically in the same order. The number of digits in the repeating period will never be greater than the divisor minus 1.	$\frac{1}{6} = \frac{1}{2 \times 3}$ $\frac{1}{6} = 0.1666\ldots$ (repeats 6) $\frac{1}{30} = \frac{1}{2 \times 3 \times 5}$ $\frac{1}{30} = 0.0333\ldots$ (repeats 3) $\frac{3}{11} = 0.2727\ldots$ (repeats 27) $\frac{3}{7} = 0.428571428571\ldots$ (repeats 428571) A bar over a set of digits means that this set of digits is repeated. $\frac{3}{11} = 0.2727 = 0.\overline{27}$ $\frac{1}{6} = 0.1666 = 0.1\overline{6}$

Mathematical Ideas | **Illustrations and Explanations**

2. OPERATIONS ON DECIMALS

Addition: Because a decimal and a fraction may be used to name the same number, addition of decimals can be developed from what is known about the addition of fractional numbers.

$$.75 + .21 = \frac{75}{100} + \frac{21}{100} = \frac{96}{100} = .96$$

Because decimals use place value notation, addition can be developed from what is known about the addition of whole numbers.

$$0.75 + 0.21 = .96 \text{ or } \begin{array}{r} .75 \\ +\ .21 \\ \hline .96 \end{array}$$

75 hundredths + 21 hundredths = 96 hundredths = .96

$$1.85 + .7 = 2.55 \text{ or } \begin{array}{r} 1.85 \\ +\ .70 \\ \hline 2.55 \end{array}$$

Subtraction: As with the addition, the subtraction of decimals may be developed from what is known about the subtraction of fractional numbers and whole numbers.

$$\begin{aligned} .75 - .6 &= (.7 + .05) - .60 \\ &= (.7 - .6) + .05 \\ &= .1 + .05 \\ &= .15 \end{aligned}$$

$$\begin{array}{r} .75 \\ -\ .60 \\ \hline .15 \end{array} \qquad \begin{array}{r} .75 \\ -\ .6 \\ \hline .15 \end{array}$$

From knowledge of place value notation and its use in addition and subtraction of decimals it may be generalized that ones' place, tenths' place, etc. are aligned in vertical form by aligning the decimal points.

$$\begin{array}{r} 6.45 \\ -\ 2.33 \\ \hline 4.12 \end{array} \qquad \begin{array}{r} \text{6 ones + 4 tenths + 5 hundredths} \\ -\ \text{2 ones + 3 tenths + 3 hundredths} \\ \hline \text{4 ones + 1 tenth + 2 hundredths} \end{array} \text{ or } 4.12$$

Multiplication: Multiplication of decimals may be developed from what is known about the multiplication of fractional numbers.

$$.3 \times .2 = \frac{3}{10} \times \frac{2}{10} = \frac{3 \times 2}{10 \times 10} = \frac{6}{100} = .06 \qquad \begin{array}{r} .3 \\ \times\ .2 \\ \hline .06 \end{array}$$

$$.8 \times .62 = \frac{8}{10} \times \frac{62}{100} = \frac{8 \times 62}{10 \times 100} = \frac{496}{1000} \qquad \begin{array}{r} .62 \\ \times\ .8 \\ \hline .496 \end{array}$$

The product equals the product of the whole numbers in the numerator divided by the product of the numbers in the denominator.

The development of a pattern showing division by powers of ten may prove useful for the understanding of the placement of the decimal point in a numeral:

Pattern:

$$\frac{1}{10} = .1$$

$$\frac{1}{100} = .01$$

$$\frac{1}{1000} = .001$$

$$\frac{1}{10000} = .0001$$

$$3.5 \times .023 = \frac{35}{10} \times \frac{23}{1000} = \frac{35 \times 23}{10 \times 1000} = \frac{805}{10{,}000} = .0805$$

$$3.5 \times .023 = .023 \times 3.5 \qquad \begin{array}{r} .023 \\ \times\ 3.5 \\ \hline .0115 \\ .069 \\ \hline .0805 \end{array} \qquad \begin{array}{r} 3.5 \\ \times\ .023 \\ \hline .0105 \\ .070 \\ \hline .0805 \end{array}$$

The product equals 23 × 35 divided by 10,000.

Division: Division of decimals may be developed from what is known about the division of fractional numbers.

$$.6 \div .3 = \frac{6}{10} \div \frac{3}{10} = 2$$

The division of decimals is simplified by using the properties of 1, $n \times 1 = n$ and $\frac{n}{n} = 1$, $n \neq 0$, to obtain an equivalent fraction.

$$.96 \div .12 = \frac{.96}{.12} \times \frac{100}{100} = \frac{.96 \times 100}{.12 \times 100} = \frac{96}{12} = 8$$

$$.96 \div .12 = .12\overline{).96} \qquad\qquad 12\overline{)96}$$

$$15 \div .125 = \frac{15}{.125} = \frac{15 \times 1000}{.125 \times 1000} = \frac{15{,}000}{125} = 120$$

$$.125\overline{)15.} \qquad\qquad 125\overline{)15{,}000}$$

Ratio and Proportion

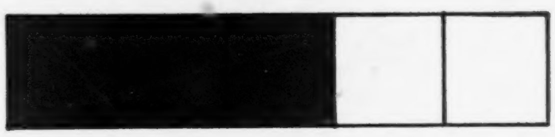
1. THE NATURE OF RATIO AND PROPORTION

Ratio: A ratio is a comparison of two numbers by division. A ratio, therefore, is a rational number. It may be expressed in the following ways:

(3, 5); $\frac{3}{5}$; 3 ÷ 5; or 3:5

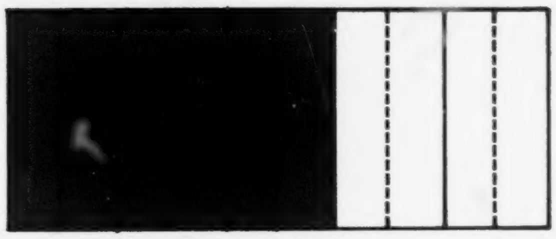
(read "3 is to 5" or "3 out of 5" or "3 per 5")

A ratio is a way to compare two numbers:



3 shaded parts to
5 congruent parts

The ordered pair (3, 5), meaning 3 out of 5, represents the ratio of the shaded portion to the entire region. The shaded portion is $\frac{3}{5}$ of the region.

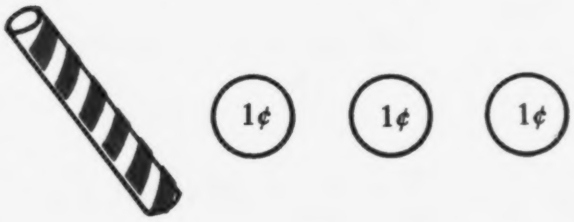


6 shaded parts to
10 congruent parts

The ordered pair (6, 10), meaning 6 out of 10, represents the ratio of the shaded portions to the entire region. The shaded portion is $\frac{6}{10}$ of the region.

Illustrated in another way:

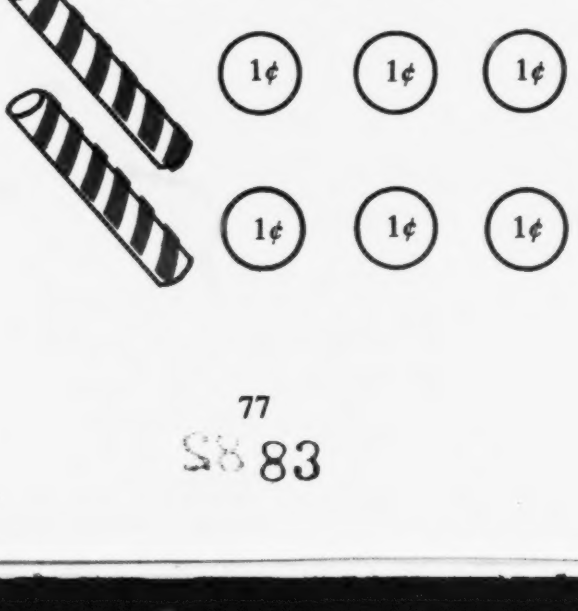
1 piece of candy costs 3¢



(1, 3) or $\frac{1}{3}$

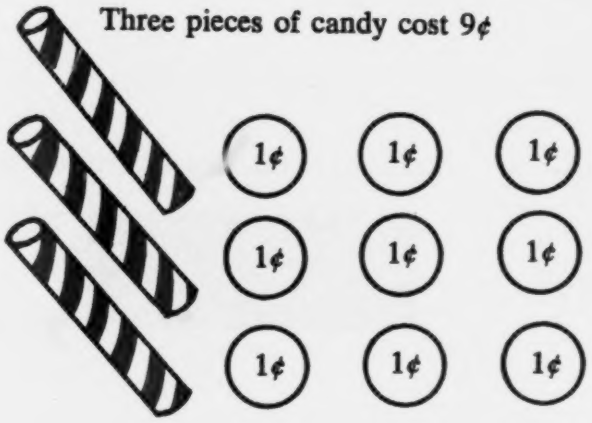
(1 out of 3)
or
(1 per 3)

Two pieces of candy cost 6¢



(2 out of 6)
or
(2, 6) or $\frac{2}{6}$ or (2 per 6)

Three pieces of candy cost 9¢



(3 out of 9)
or
(3, 9) or $\frac{3}{9}$ or (3 per 9)

Proportion: A proportion is a statement that two or more ratios are equal.

As shown in the illustrations above:

$$\frac{3}{5} = \frac{6}{10} \text{ and } \frac{1}{3} = \frac{2}{6} = \frac{3}{9}$$

$\frac{a}{b} = \frac{c}{d}$ is read "*a* is to *b* as *c* is to *d*"

2. PROPERTIES AND OPERATIONS

Since a ratio is a rational number, the operations and properties for rational numbers apply to ratios.

Percent

1. NATURE OF PERCENT

A percent is a special ratio which always has 100 as its denominator. 100, therefore, is always the basis for comparison.

10% means 10 per hundred, expressed as (10, 100) or $\frac{10}{100}$

15% means 15 per hundred, expressed as (15, 100) or $\frac{15}{100}$

In general, the rational number $\frac{a}{b}$ can be expressed as percent by finding the number *c* in the statement, $\frac{a}{b} = \frac{c}{100}$

$\frac{a}{b} = \frac{c}{100}$ is interpreted: *a* is *c* percent of *b* or *a* per *b* is the same as *c* per 100, or *a* is to *b* as *c* is to 100.

2. OPERATIONS WITH PERCENT

Solving problems using percent.

Example:

3 is what % of 8?

3 = c% of 8

Other ways of stating this are:

$\frac{3}{8}$ is what percent?

3 out of 8 is the same as *c* out of 100

$$\frac{3}{8} = \frac{c}{100}$$

$3 \times 100 = 8 \times c$ **definition of equivalent fractions**

$\frac{1}{8} \times 8 \times c = 3 \times 100 \times \frac{1}{8}$ **multiplication by the reciprocal of 8**

$$c = \frac{3 \times 100}{8} = \frac{300}{8} = 37\frac{1}{2} = 37.5$$

$$\frac{3}{8} = \frac{37.5}{100} = 37.5\%$$

This shows that 3 is 37.5% of 8.

Example: Bob sold 50% of his 12 rabbits.
How many rabbits did he sell?

Using the fundamental statement:

***a* out of *b* is the same as *c* out of 100, $\frac{a}{b} = \frac{c}{100}$**
***a* out of 12 is the same as 50 out of 100**

Two ways of solving the proportion are:

$\frac{a}{12} = \frac{50}{100}$ **or** $\frac{a}{12} = .5$

$a = 12 \times \frac{50}{100}$ $a = 12 \times .5$

$a = \frac{600}{100}$ $a = 6.0$

$a = 6$

6 out of 12 is the same as 50 out of 100.

Bob sold 6 rabbits.

Example: Hal sold 6 rabbits. This was 50% of the rabbits he owned. How many rabbits did he have before selling any?

6 out of b = 50 out of 100

$$\frac{6}{b} = \frac{50}{100} \qquad \text{or} \qquad \frac{6}{b} = .5$$

$$\begin{aligned} 6 \times 100 &= 50 \times b \\ 600 &= 50 \times b \\ \frac{600}{50} &= b \\ 12 &= b \end{aligned} \qquad\qquad \begin{aligned} 6 &= .5 \times b \\ .5 \times b &= 6 \\ b &= \frac{6}{.5} \\ b &= 12 \end{aligned}$$

6 out of 12 is the same as 50 out of 100.

Hal had 12 rabbits before selling any.

Example: Leonard paid 6% interest for the year on $400 he borrowed. How much interest did he pay?

$$\frac{a}{400} = \frac{6}{100}$$

He paid $6 for $100, therefore, 4 × 6 for $400 or $24. Leonard paid $24 interest.

Find the number of hundreds (in this case 4), and multiply by the number per hundred (in this case 6), 4 × 6 = 24.

In general, any problem involving percent can be solved by utilizing:

1. The definition of ratio as a rational number
2. Properties of rational numbers
3. Interpretation of the statement as the proportion:

$$\frac{a}{b} = \frac{c}{100}$$ or *a* per *b* is the same as *c* per 100.

Since the proportion given in 3 above may be used to solve any problem involving percent, the task becomes one of writing an appropriate proportion.

Properties for Four Arithmetic Operations for Rational Numbers

To this point, the following sets of numbers have been introduced:

1. The counting numbers
2. The whole numbers
3. The fractional numbers
4. The integers
5. The negatives of the fractional numbers.

With each set of numbers, the four operations of arithmetic and certain properties of each set of numbers with respect to these operations have been introduced.

The sets of numbers mentioned above include all numbers studied in the elementary school.

These sets of numbers make up the set of rational numbers. The set of rational numbers is defined as the set of all numbers which can be expressed in the form $\frac{a}{b}$ where *a* and *b* are integers, except $b \neq 0$. Since each of the above mentioned sets of numbers satisfies this definition, each is a subset of the set of rational numbers.

Listed below are properties for each of the four arithmetic operations with respect to the set of rational numbers—*a*, *b*, and *c* being rational numbers:

Addition

1. Closure $a + b$ is a rational number
2. Commutativity $a + b = b + a$
3. Associativity $(a + b) + c = a + (b + c)$
4. Identity Element (Zero) $a + 0 = 0 + a = a$
5. Additive Inverse $a + (^-a) = 0$
   (opposite)

Subtraction

1. Closure $a - b$ is a rational number
2. Not Commutative $a - b \neq b - a$
3. Not Associative $(a - b) - c \neq a - (b - c)$
4. Identity Element (Zero on the right)
   $a - 0 = a$, but $0 - a \neq a$

Multiplication

1. Closure $a \times b$ is a rational number
2. Commutativity $a \times b = b \times a$
3. Associativity $(a \times b) \times c = a \times (b \times c)$
4. Identity Element (One) $a \times 1 = 1 \times a = a$
5. Multiplication property of zero
   $a \times 0 = 0 \times a = 0$
6. Reciprocal (Multiplicative Inverse)
   $a \times \frac{1}{a} = \frac{1}{a} \times a = 1$
7. Distributivity over addition:
   $a \times (b + c) = (a \times b) + (a \times c)$

   over subtraction:
   $a \times (b - c) = (a \times b) - (a \times c), b > c$

Division

1. Closure $a \div b$ is a rational number
2. Not Commutative $a \div b \neq b \div a$
3. Not Associative $(a \div b) \div c \neq a \div (b \div c)$
4. Identity Element (One on the right)
   $a \div 1 = a$, but $1 \div a \neq a$
5. Distributivity over addition (Distribution on dividend *only*):
   $(b + c) \div a = (b \div a) + (c \div a)$

   over subtraction:
   $(b - c) \div a = (b \div a) - (c \div a)$

TEACHING SUGGESTIONS FOR RATIONAL NUMBERS

Fractional Numbers

1. NATURE OF FRACTIONAL NUMBERS

Young children begin to develop ideas of fractions as parts of wholes as they share halves of apples, cookies or candy bars. The idea that halves of objects may have many sizes and shapes grows as they divide different kinds of objects. This understanding may be applied to fractional numbers in the primary grades through paper folding. Each child may fold a sheet of paper into two parts of the same size and color one part blue. Such questions as the following should be used:

a. How many parts are colored blue?

b. How many parts are there in all?

The fractional number $\frac{1}{2}$ should be recorded as the point is made that the 1 tells the number of blue parts and the 2 tells the number of parts in all.

This procedure should be repeated with fourths: folding the paper into two equal parts and again into four equal parts, coloring one part red and two parts green and using such questions as the following:

a. How many parts are colored red?

b. How many parts are colored green?

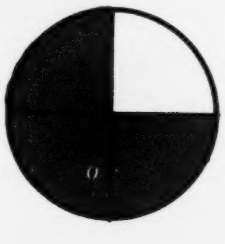
c. How many parts are colored red and green?

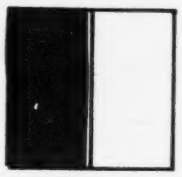
d. How many parts are not colored?

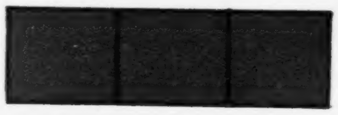
e. How many parts are there in all?

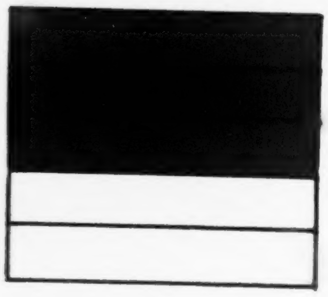
As the fractional number $\frac{1}{4}$ is recorded, emphasis is again given to the idea that the 1 tells the number of red parts and that 4 tells the number of parts in all. As $\frac{2}{4}$ is recorded, children can see that 2 tells the number of green parts and 4 the number of parts in all. As understanding of the meaning of fractional number grows, children will discover that $\frac{1}{2}$ and $\frac{2}{4}$ represent the same fractional part of the entire sheet of paper.

Models such as the following may be used:


A


B


C


D

As parts are discussed, numbers may be recorded on charts as follows:

	A	B	C	D
Number of shaded parts	3	1	3	3
Number of parts in all	4	2	3	5

	A	B	C	D
Number parts not shaded	1	1	0	2
Number parts in all	4	2	3	5

Older children may write the ordered pairs as (3, 4). Considering fractional numbers as ordered pairs is not difficult for children if the idea is connected with a physical model. For example: Using this model, asking questions similar to those above, recording the ordered pair as (3, 4) and stressing that the first number of the ordered pair always tells the number of parts of the same size that are considered (shaded) and the second number tells the total number of parts of the same size.

Using other models of fractional numbers children may be asked to give the fractional number represented and to write its name as a fraction and as an ordered pair.

When using physical models for developing understanding of rational numbers, it is important to use different kinds of models: Regions (circular, square, and rectangular), segments, and sets of objects. In this way children will not get the notion that fractional numbers are just parts, but that they are numbers.

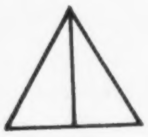
Each child should have his own kit of fractional models including wholes, halves, fourths, eighths, thirds, sixths, and a model of a number line to use at appropriate grade levels. These may be made by children.

A number line can be used in developing understanding of the nature of fractional numbers. Starting with a number line divided into unit segments, children may see how a unit segment may be divided into parts of the same size and that fractions may be associated with points on the number line. Number lines showing halves, fourths, eighths, etc. may be made as needed.

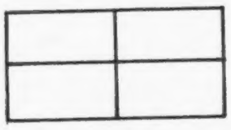
Word problems using fractional numbers may be used. For example: Mother cut a cake into eight equal pieces. She gave me 1 piece. What part of the cake did she give me? What part of the cake is she keeping for dinner?

Exercises of the following types may aid in developing understanding of fractional numbers:

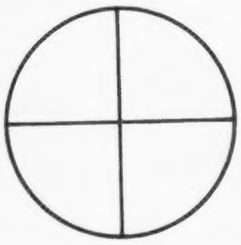
a. Shade the part of the figure that shows what the fraction means:



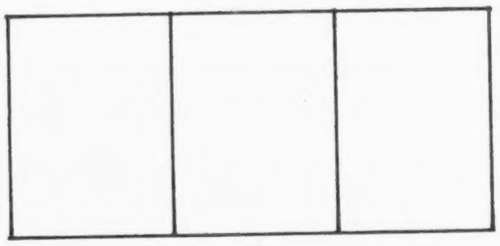
$\frac{1}{2}$



$\frac{3}{4}$

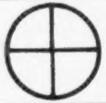
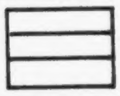
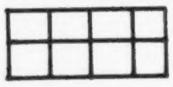


$\frac{2}{4}$

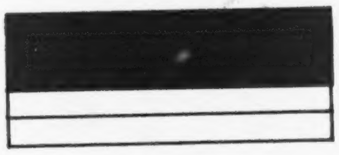


$\frac{2}{3}$

b.

	color parts named by:
	(1, 4)
	(3, 3)
	(5, 8)

c.

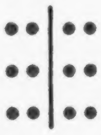


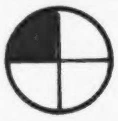
What does the number pair (2, 5) tell about the picture?

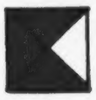
What does the number pair (3, 5) tell about the picture?

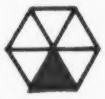
What does the number pair (5, 5) tell about the picture?

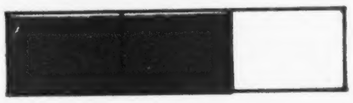
d. Match the ordered pair with the correct pictures: (1, 4); (1, 6); (2, 3); (6, 12); (3, 4)



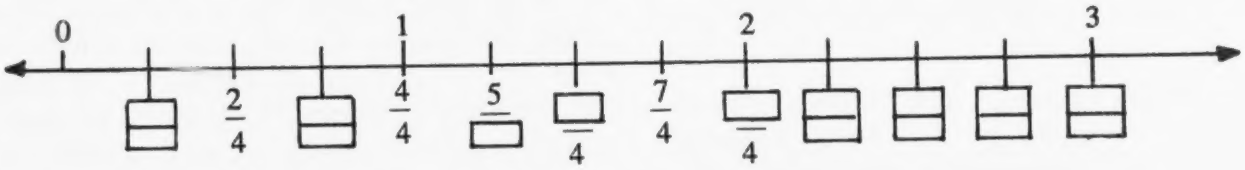




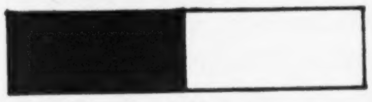




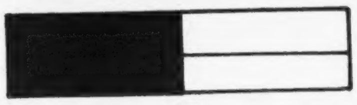
e. Add labels where omitted on the following number line:



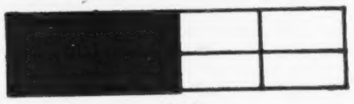
Equivalent fractions: Congruent models should be used to help children discover that there are many names for the same fractional number. For example:



(a)



(b)



(c)

Such questions as the following can aid in the discovery:

For model (a):

How many parts are shaded?

How many parts are there in all?

What fractional number is illustrated by this model? (At this point the fractional number should be recorded on the chalkboard as (1, 2) or $\frac{1}{2}$).

The same questions should be repeated for model (b) and the fractional number recorded as (2, 4) or $\frac{2}{4}$, and for model (c) and the number recorded as (4, 8) or $\frac{4}{8}$.

When this has been done, the following questions should be used:

Does each fractional number represent the same amount of space in the model?

Are $\frac{1}{2}, \frac{2}{4}, \frac{4}{8}$ names for the same fractional number?

Are there other names for this number?

Can you demonstrate or prove this with a model?

This development should be repeated with other fractional numbers, such as, $\frac{1}{3}, \frac{2}{6}, \frac{3}{9}$, or $\frac{1}{5}, \frac{2}{10}, \frac{3}{15}$, etc. using different types of models, both concrete and semi-concrete.

A number line may be marked to show halves, fourths, and eighths. Children may use this to show other names for segments of the line labeled $\frac{1}{2}$ and $\frac{1}{4}$.

Attention may be directed to the classroom where physical models representing equivalent fractional numbers may be seen: windowpanes, panels, floor tiles.

After many experiences with physical models, children can be led to discover that an equivalent fraction can be obtained by multiplying or dividing the numerator and denominator by the same counting number. For example: A chart similar to the one in column 2 pages 53-54 (Rational Numbers) may be built with children to show 2 units divided into halves, fourths, eighths, sixteenths . . . As points are labeled and the chart is more complete, children discover that a fractional number can be named by an infinite number of ordered pairs.

In the beginning, children should use models to find equivalent fractions. Once the notion is developed, charts similar to those on page 53 may be placed on bulletin boards for reference. Charts showing halves, fourths, eighths, sixteenths . . . ; charts of halves, thirds, sixths, twelfths . . . ; charts of halves, fifths, tenths . . . should be included.

Preliminary work with factoring numbers, finding common factors, and finding equivalent fractions is necessary before introducing the idea of fractions in *simplest form.* When introducing "simplest form," such questions as the following may be used:

What common factors do the terms of $\frac{8}{16}$ have?

What is the greatest common factor of 8 and 16?

What is the result when you divide the numerator and denominator of $\frac{8}{16}$ by the greatest common factors of 8 and 16?

Are $\frac{8}{16}$ and $\frac{1}{2}$ equivalent fractions?

Do the terms of $\frac{1}{2}$ have any common factors other than 1?

It should be explained that $\frac{1}{2}$ is a fraction in simplest form because the numerator and denominator have no common factors other than 1. This procedure should be followed with other examples including some fractions whose numerators will not be one in simplest form and some fractions already in simplest form.

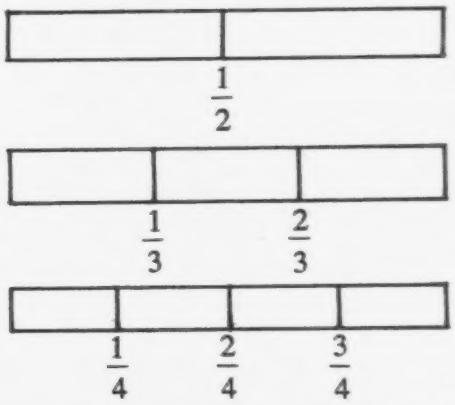
After many experiences with physical models, children are able to discover that an equivalent fraction can be obtained by multiplying or dividing the numerator and denominator by the same counting number. For example, by referring to models they can understand why: $\frac{2}{4} = \frac{2 \times 1}{2 \times 2}$; $\frac{4}{8} = \frac{4 \div 4}{8 \div 4} = \frac{1}{2}$, etc.

Order of fractional numbers: The number line may be used to develop understanding of the order of fractional numbers. The idea may be introduced on a large demonstration number line similar to the one on page 56, with points located but unlabeled. When points have been properly labeled, questions like the following may be used to lead children to use the number line to discover the order of fractional numbers:

> Which segment on the number line is longer, the one associated with $\frac{1}{2}$ or the one associated with $\frac{1}{4}$?
>
> . . . the one associated with $\frac{1}{4}$ or the one associated with $\frac{1}{3}$?
>
> . . . the one associated with $\frac{2}{3}$ or the one associated with $\frac{3}{4}$?
>
> Which are true: $\frac{1}{2} > \frac{1}{4}$; $\frac{1}{4} > \frac{1}{3}$; $\frac{2}{3} > \frac{3}{4}$?

Children may use a strip of paper of a unit length which may be folded to show halves, another strip of paper (same unit length) which may be folded to show fourths, and another strip of paper (same unit length) which may be folded to show thirds.

These may be compared to show order of fractional numbers.



After comparing many pairs of fractions in this way, children should be able to generalize that a given fractional number on a number line is less than the one to its right and greater than the one to its left.

Physical models may also be used to compare unlike fractions with numerators of 1. Models similar to that below may be used with questions.

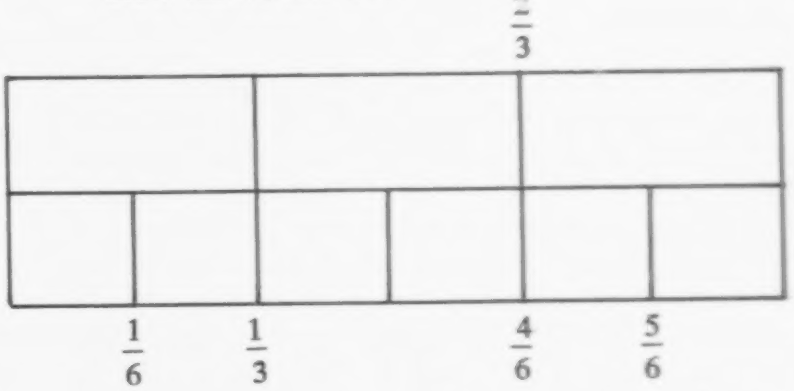
Which is larger, $\frac{1}{3}$ or $\frac{1}{6}$? After working with many models of different unit fractions, children should be able to generalize:

> As the number of congruent parts of a whole increases, the size of each part decreases.
>
> As the number of congruent parts of a whole decreases, the size of each part increases.

After comparing unit fractions, children are ready to compare unlike fractions in which one fraction needs renaming.

Children may be asked such questions as:

> Which is larger $\frac{2}{3}$ or $\frac{5}{6}$?
>
> How can we find out?

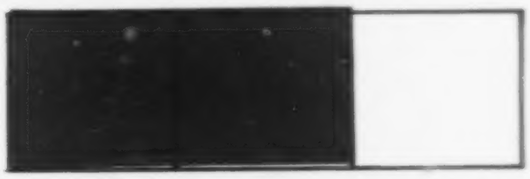


Models similar to those above may be used and comparisons made. As understanding develops that this is a process of renaming, children may be led to compare pairs of fractional numbers by renaming and then verifying answers with models.

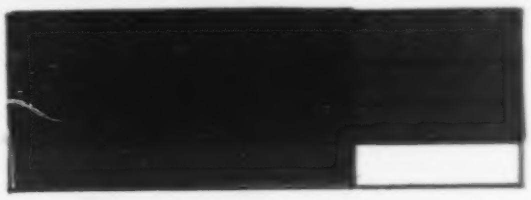
To compare fractions which require renaming of both fractional numbers, children may make models.

For example: Models of $\frac{2}{3}$ and $\frac{3}{4}$ may be compared by changing each to twelfths as follows:

Draw a unit rectangle. Draw vertical segments to show thirds. Color two sections, (step 1).



Draw horizontal segments to show fourths. Use a contrasting color to shade three of these new sections, (step 2).



Answer these questions:

> How many sections are blue? 8
>
> How many small sections are red? 9

How many small sections are there? 12

$\frac{8}{12}$ are blue $\frac{9}{12}$ are red. $\frac{8}{12} = \frac{2}{3}$ and $\frac{9}{12} = \frac{3}{4}$

then $\frac{3}{4} > \frac{2}{3}$.

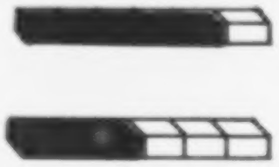
Transparencies of such models can be made by children. The larger fraction may be readily determined by comparing the number of twelfths. Other pairs of fractional numbers may then be compared without using models and the answers verified by drawings if needed.

Density of fractional numbers: As the density of rational numbers is difficult for children to understand, it is taught much later than are other fractional number concepts. The number line may be used by asking appropriate questions. Children may discover that a fractional number can always be found between two given fractional numbers. This may be demonstrated by associating fractions with points on the number line.

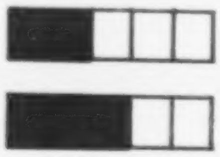
For example: between $\frac{1}{4}$ and $\frac{1}{2}$ is found $\frac{3}{8}$, between $\frac{1}{4}$ and $\frac{3}{8}$ is found $\frac{5}{16}$, etc. When the physical model gets in the way and it becomes difficult to determine these points, the discussion may be carried on by continuing to name fractions between points. If a question like, "Do you think it is possible to name all the fractional numbers between two given fractional numbers?" brings the answer "yes," the child should try to prove it with examples. Someone who answers "no" may attempt to justify his point of view. With experiences such as these, children will be able to make the generalization regarding the density of fractional numbers; i.e., between any two fractional numbers there is another fractional number.

2. Operations on Fractional Numbers

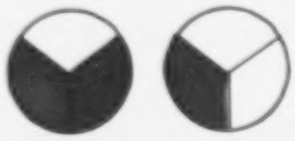
Addition: Physical models should be used by children for concrete experience in adding fractional numbers with like denominators. Sets of cubes, strips of paper, or rectangular and circular regions to represent fractional parts of a unit, may all be used to develop understanding of addition with like fractions.



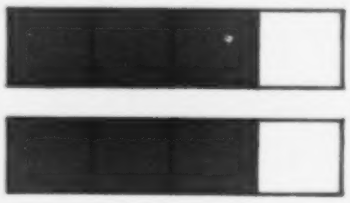
$\frac{5}{6} + \frac{3}{6} = \square$



$\frac{2}{5} + \frac{3}{5} = \square$



$\frac{2}{3} + \frac{1}{3} = \square$

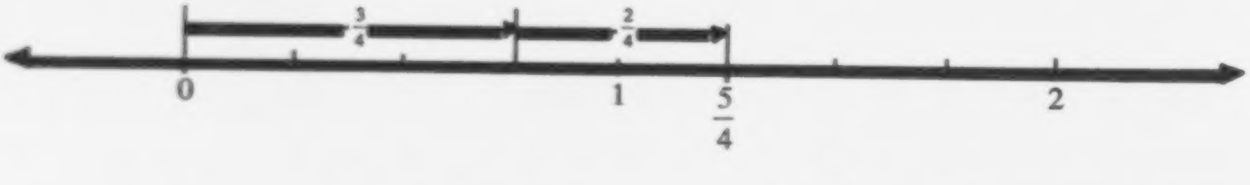


$\frac{3}{4} + \frac{3}{4} = \square$

These experiences should be discussed as mathematical sentences for the addition of the fractional numbers are written on chalkboard or overhead projector. When enough experiences of this type have been used to develop understanding of the idea, sums may be found for addition examples without using manipulative materials. Generalization about the addition with like fractions may then be expressed.

The number line may also be used to find the sum of fractional numbers named with like denominators.

For example:



$\frac{3}{4} + \frac{2}{4} = \square$

Children who have had many experiences with equivalent fractions should have little need for physical models in adding fractions with unlike denominators. They should be able to generalize that it is necessary to rename these fractional numbers in order to find the sum. If there is need for use of physical models, it is clear that understanding of equivalent fractions is not adequate and this should be the point of emphasis before practice of addition.

Subtraction: Physical models similar to those used in addition of fractional numbers may be used to develop understanding of subtraction with like fractions. As $\frac{1}{3}$ is subtracted from $\frac{2}{3}$ or $\frac{5}{6}$ from $\frac{10}{6}$, using fractional parts of unit models, children can be led to express the generalization. If understanding of equivalent fractions and of the addition with unlike denominators is complete, subtraction with unlike fractions by rewriting them with the least common denominators should present few problems.

Children should have opportunity to solve number sentences written in several forms:

$$\frac{4}{5} - \frac{2}{5} = \square \qquad \frac{2}{3} + \square = \frac{7}{3} \qquad \square + \frac{2}{5} = \frac{2}{3}$$

They should have experience in writing number sentences to solve number stories. For example: With $\frac{7}{8}$ yards of ribbon on a spool, does Mary have enough to make a bow if the pattern calls for $\frac{5}{6}$ yards?

Will there be any ribbon left over?

$\frac{7}{8} - \frac{5}{6} = \frac{21}{24} - \frac{20}{24} = \frac{1}{24}$ Yes. There is $\frac{1}{24}$ of a yard left over.

Pupils may create number stories to fit mathematical sentences. For example: $\frac{9}{16} - \square = \frac{2}{16}$

How much ribbon was used if Jane had $\frac{2}{16}$ of a yard left from a roll which contained $\frac{9}{16}$ of a yard?

Bill lives $\frac{9}{16}$ of a mile from school. Joe lives $\frac{2}{16}$ of a mile from school. How far apart are their houses?

If children experience difficulty in doing this, it may indicate that understanding of ideas involved in subtraction is not adequate.

Multiplication: In order to capitalize on children's understanding of "of" in such mathematical sentences as $\frac{1}{2}$ of $8 = 4$, teachers may introduce multiplication of fractional numbers by listing on the chalkboard such mathematical sentences as the following, using accompanying questions to draw attention to the pattern as it develops:

$\frac{1}{2}$ of $32 = 16$ $\qquad 4 \times 8 = 32$

$\frac{1}{2}$ of $16 = 8$ $\qquad 2 \times 8 = 16$

$\frac{1}{2}$ of $8 = 4$ $\qquad 1 \times 8 = 8$

$\frac{1}{2}$ of $4 = 2$ $\qquad \frac{1}{2} \times 8 = 4$

$\frac{1}{2}$ of $2 = 1$ $\qquad \frac{1}{4} \times 8 = 2$

$\frac{1}{2}$ of $1 = \frac{1}{2}$ $\qquad \frac{1}{8} \times 8 = 1$

$\frac{1}{2}$ of $\frac{1}{2} = \frac{1}{4}$

$\frac{1}{2}$ of $\frac{1}{4} = \frac{1}{8}$

What happens to the product as one factor is halved?

Does this also happen when 1 is the factor that is halved?

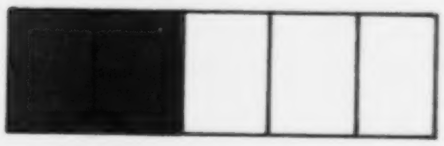
Does this happen when the factor halved is less than 1?

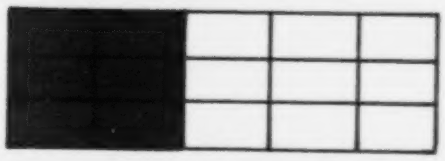
Is the same thing true when the multiplication sign is used instead of "of"?

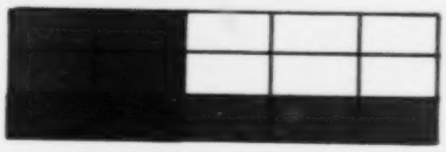
Does the same pattern develop for other sets of numbers?

The "of" relationship may also be used with physical models to develop the multiplication algorithm.

To develop understanding of the mathematical sentence $\frac{1}{3}$ of $\frac{2}{5} = \square$ a rectangular region similar to following may be drawn on the chalkboard and questions used to involve children in the process of finding $\frac{1}{3}$ of $\frac{2}{5}$ or $\frac{1}{3} \times \frac{2}{5}$ of the region:







Why is it necessary to split or separate the region into five equal parts?

How many fifths should be colored?

Why is it also necessary to split the region into three equal parts?

How many thirds should be colored?

Into how many parts is the region now separated?

How many of these fifteenths have been colored twice?

What is $\frac{1}{3}$ of $\frac{2}{5}$? Complete the mathematical sentence: $\frac{1}{3}$ of $\frac{2}{5} = \square$ $\quad \frac{1}{3} \times \frac{2}{5} = \square$

Understanding of multiplication of fractional numbers using a rectangular region may also be developed through the use of transparencies with overlays. Physical models should be used until understanding is developed. Children may then make their own drawings to illustrate other mathematical sentences.

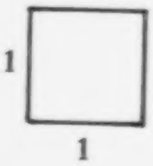
The area of a rectangular region may also be used to illustrate the algorithm for the multiplication of fractional numbers. A region whose side measures are whole numbers should be reviewed through the use of physical models.

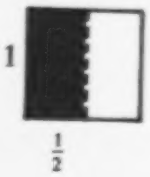
Questions may help to recall the fact that the area is found by multiplying the number of rows by the number of square units in each row. Next a square region with area 1 and with side measures of 1 unit may be displayed and discussed:

What are the dimensions of the region which is shaded?

What is the area of the shaded region?

Is the sentence, $\frac{1}{2} \times 1 = \frac{1}{2}$ true? Why?





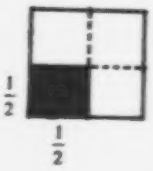
Using other models, repeat the above questions:

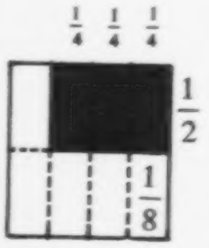
What are the dimensions of the region which is shaded?

What is the area of the shaded region?

Is the sentence $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ true? Why?

Is the sentence $\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$ true? Why?





$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Shaded area is $\frac{1}{4}$ of unit.

$\frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$

Shaded area is $\frac{3}{8}$ of unit.

After many experiences of this type children should be able to generalize that to find the product of fractional numbers, the numerators are multiplied, the denominators are multiplied and the product is written as a fraction. For example: $\frac{3}{5} \times \frac{2}{7} = \frac{3 \times 2}{5 \times 7} = \frac{6}{35}$

Division: Division of fractional numbers may be approached in more than one way, three of which are outlined below. Teachers should select one approach at a time to develop, discussing procedures step by step. When ideas are understood to the point where the operation can be performed, another approach may be demonstrated in the same way to reinforce understanding of the principle that division by a fractional number is the same as multiplication by its reciprocal. This is particularly important when either the complex fraction or inverse operation is the first approach used. When the general principle is accepted children should be urged to use the algorithm that they understand best. Before teaching division of fractional numbers by the complex fraction algorithm, children must understand the following: (See Rational Numbers, pages 61-62.

1. $6 \div 3 = \frac{6}{3}$; $2 \div 4 = \frac{2}{4}$; $5 \div 1 = \frac{5}{1}$; $a \div b = \frac{a}{b}$

2. $\begin{array}{r}1\\7\overline{)7}\end{array}$; $7 \div 7 = 1$; $\frac{7}{7} = 1$; $\frac{\frac{1}{2}}{\frac{1}{2}} = 1$; $\frac{\frac{5}{6}}{\frac{5}{6}} = 1$; $\frac{\frac{a}{b}}{\frac{a}{b}} = 1$

3. $16 \times 1 = 16$; $\frac{1}{3} \times 1 = \frac{1}{3}$; $1 \times \frac{4}{3} = \frac{4}{3}$; $\frac{a}{b} \times 1 = \frac{a}{b}$

4. $\begin{array}{r}8\\1\overline{)8}\end{array}$; $\frac{8}{1} = 8$; $8 \div 1 = 8$; $a \div 1 = a$

5. $\frac{2}{3} \times \frac{3}{2} = 1$; $\frac{1}{5} \times \frac{5}{1} = 1$; $5 \times \frac{1}{5} = 1$

6. $\frac{5}{6} \times \frac{5}{7} = \frac{5 \times 5}{6 \times 7} = \frac{25}{42}$; $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$

7. $\frac{11}{3} \div \frac{2}{7} = \frac{\frac{11}{3}}{\frac{2}{7}}$; $\frac{a}{b} \div \frac{c}{b} = \frac{\frac{a}{b}}{\frac{c}{d}}$

It must be understood that the number one has many forms which are useful in working with complex fractions. The numerator and denominator of a complex fraction can be multiplied by one in any of its forms without changing its value in the same way that any fractional number may be multiplied by one. For example:

$$\frac{\frac{5}{7} \times 1}{\frac{2}{3} \times 1} = \frac{\frac{5}{7}}{\frac{2}{3}} \quad ; \quad \frac{\frac{5}{7} \times \frac{3}{3}}{\frac{2}{3} \times \frac{3}{3}} = \frac{\frac{15}{21}}{\frac{6}{9}} = \frac{\frac{5}{7}}{\frac{2}{3}} \quad ;$$

$$\frac{\frac{5}{7} \times \frac{3}{3}}{\frac{2}{3} \times \frac{7}{7}} = \frac{\frac{15}{21}}{\frac{14}{21}} = \frac{\frac{5}{7}}{\frac{2}{3}} \quad ; \quad \frac{\frac{a}{b} \times 1}{\frac{c}{d} \times 1} = \frac{\frac{a}{b}}{\frac{c}{d}}$$

If children have difficulty understanding that a fraction is *one* number, not *two,* or that division of fractional numbers involves *two* numbers, not *four,* the following idea may be helpful:

$\frac{2}{3} \div \frac{1}{8} =$ Let $\frac{2}{3} = x$ and $\frac{1}{8} = y$

then $\frac{2}{3} \div \frac{1}{8} = x \div y = \frac{x}{y}$ and $\frac{2}{3} \div \frac{1}{8} = \frac{\frac{2}{3}}{\frac{1}{8}}$

The following questions may be used to check for these understandings:

In how many ways can the operation of dividing four by three be shown?

$\begin{array}{r}\square\\3\overline{)4}\end{array}$ $4 \div 3 = \square$ $\frac{4}{3} = \square$

What does the bar between the two whole numbers in a fraction mean?

What do the following expressions mean:

$\frac{2}{4}$, $\frac{4}{2}$, $\frac{\frac{1}{2}}{\frac{3}{4}}$, $\frac{\frac{11}{12}}{\frac{2}{2}}$

By what number may any number be multiplied without changing its value? Give examples to prove it.

By what number must a number be multiplied to produce a product of 1? Give examples to prove it.

In how many different ways may the following complex fraction be expressed:

$$\frac{\frac{7}{8}}{\frac{3}{4}}\ ?$$

$\frac{7}{8} \div \frac{3}{4}$ $\qquad$ $\square \times \frac{3}{4} = \frac{7}{8}$ $\qquad$ $\frac{3}{4} \times \square = \frac{7}{8}$

Does $\frac{2}{3} \div \frac{1}{8} = \frac{\frac{2}{3}}{\frac{1}{8}}$?

Does $\frac{2}{3} \div \frac{1}{8} = \frac{2 \div 1}{3 \div 8} = \frac{\frac{2}{1}}{\frac{3}{8}}$?

Does $\frac{\frac{2}{3}}{\frac{1}{8}} = \frac{\frac{2}{1}}{\frac{3}{8}}$?

In how many ways may 1 be expressed?

$$1,\ \frac{1}{1},\ \frac{4}{4},\ \frac{16}{16},\ \frac{203}{203},\ \frac{\frac{2}{3}}{\frac{2}{3}}\ \ldots$$

If it is clear that children understand basic ideas expressed above, the complex fraction method may be introduced, developed step by step and discussed as follows:

$\frac{3}{4} \div \frac{5}{6} = \frac{\frac{3}{4}}{\frac{5}{6}}$ Is this true? Why?

$= \frac{\frac{3}{4} \times \frac{6}{5}}{\frac{5}{6} \times \frac{6}{5}}$ Why may this operation be performed?

(3) $N \times 1 = N$
(5) reciprocal
(2) $\frac{N}{N} = 1$

$= \frac{\frac{3}{4} \times \frac{6}{5}}{1}$ Why? $\qquad$ (5) reciprocal

$= \frac{3}{4} \times \frac{6}{5}$ Why? $\qquad$ (2) $\frac{N}{N} = 1$

$= \frac{18}{20}$ Why? $\qquad$ (6) $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$

$= \frac{9}{10}$ Why? $\qquad$ (3), (2) factored

When the process is understood, pupils should be able to generalize that division by a fractional number is the same as multiplication by its reciprocal.

One algorithm for the division of fractional numbers may be developed through children's familiarity with common denominators as used in addition and subtraction. This approach can provide a logical introduction to division of fractional numbers.

Mathematical sentences similar to the following may be written on the chalkboard, and procedures discussed as each step of the solution is performed:

$\frac{1}{2} \div \frac{1}{4} = \square$

$\frac{1}{2} \times \frac{2}{2} = \frac{2}{4}$ l.c.m. of 2 and 4 is 4.

$\frac{2}{4} \div \frac{1}{4} = \frac{2 \div 1}{4 \div 4} = \frac{2 \div 1}{1} = \frac{2}{1} = 2$

$\frac{5}{6} \div \frac{7}{8}$ $\qquad$ $\frac{5}{6} \times \frac{4}{4} = \frac{20}{24}$ and $\frac{7}{8} \times \frac{3}{3} = \frac{21}{24}$
l.c.m. of 6 and 8 is 24.

$\frac{20}{24} \div \frac{21}{24} = \frac{20 \div 21}{24 \div 24} = \frac{20 \div 21}{1} = \frac{20}{21}$

If children experience difficulty in using this approach, it will indicate that they do not understand one or more of the following:

How to find the common denominator

That a number divided by itself has 1 as the quotient

How to express the numerator of the fraction $\frac{20 \div 21}{1}$ as a quotient.

Attention should be given to the area of difficulty before practice on the algorithm.

Familiarity with division of whole numbers as the inverse of multiplication may lead some children to ask if this relationship holds true for fractional numbers. As the procedure is developed and the relationship understood, further evidence can be seen that division by a fractional number is the same as multiplication by its reciprocal.

Before developing the algorithm, children must have clear understanding of statements given under the complex fraction approach, page 63, especially statements 5 and 8. They *must* recognize division as the inverse of multiplication—they must understand that if $3 \times 4 = 12$, then $12 \div 3 = 4$ and $12 \div 4 = 3$; that $\frac{2}{3} \div \frac{1}{7}$ means that $\frac{1}{7} \times \square = \frac{2}{3}$; that $a \times b = c$ means $\frac{c}{b} = a$, $(c \div b = a)$ and that $\frac{c}{a} = b$, $(c \div a = b)$.

Questions similar to those used to determine readiness for the complex fraction approach, page 89 should be used to review understanding of these basic ideas.

A problem may then be developed through steps as outlined on page 63. The process must be discussed step by step with children supplying reasons for each operation. Questions should aid in making the process clear.

For example:

$\frac{5}{6} \div \frac{3}{8} = \square$ and $\square \times \frac{3}{8} = \frac{5}{6}$

Is this true? Why?

Multiplication is the inverse of division

$\square$ must be $\left(\frac{5}{6} \times \triangle\right)$

$\left(\frac{5}{6} \times \triangle\right) \times \frac{3}{8} = \frac{5}{6}$ Why? Substitution

$\frac{5}{6} \times \left(\triangle \times \frac{3}{8}\right) = \frac{5}{6}$ Is this also true? Why?

Associative property

If $\frac{5}{6} \times (\quad) = \frac{5}{6}$,
what must be in ( )? (1) Why? $N \times 1 = N$

Then $\triangle \times \frac{3}{8} = 1$ True? Why?

Then $\triangle = \frac{8}{3}$ Why? Reciprocal

$\square = \frac{5}{6} \times \frac{8}{3}$ or $\frac{5}{6} \times \frac{8}{3} = \square$ Why?

$\left(\frac{5}{6} \times \frac{8}{3}\right) \times \frac{3}{8} = \frac{5}{6}$ True? Why? Substitution

$\frac{5}{6} \div \frac{3}{8} = \frac{5}{6} \times \frac{8}{3}$ Why?

The idea that division by a fractional number is the same as multiplication by its reciprocal should again be evident. Most children will accept this as a principle to be trusted and be ready to use it as the reason for a simple, direct approach to division of fractional numbers.

Estimation should be used to determine reasonableness of answers. This must be done through conversation; it should not be done through seat work. Discussion of such questions as the following may be helpful:

> Which quotient is greater, $12 \div 2$ or $12 \div \frac{1}{2}$? Why? $18 \div \frac{1}{3}$ or $18 \div \frac{2}{3}$? Why? Can physical models be used to prove the above?
>
> Are the following true or false?
>
> $\frac{3}{4} \div \frac{1}{2} > \frac{1}{4} \div \frac{1}{2}$ (T)
>
> $\frac{7}{3} \div \frac{1}{7} < 2 \div \frac{1}{7}$ (F)

Before practice for mastery there should be opportunity to discuss and solve such sentences as the following:

$\frac{1}{5} \div \frac{3}{8} = N \qquad \frac{2}{5} \div N = \frac{1}{2} \qquad N \div \frac{3}{4} = \frac{7}{2}$

Opportunity should be provided for pupils to write, discuss and solve practical problems involving division of fractional numbers.

Properties of Addition and Subtraction of Fractional Numbers

What children know about the properties of whole numbers may be used to determine whether these properties also hold for fractional numbers. Such activities as the following may be used:

1. Finding the sum of fractional numbers and discussing the set of numbers to which the sum belongs to enable children to see that the set of rational numbers is closed for addition.

2. Finding the difference of several pairs of fractional numbers to discover that, using numbers which are familiar at this point, it is not always possible to subtract. It becomes apparent that subtraction is not closed for the set of fractional numbers.

3. Using the number line to see that addition is commutative and associative:

$$\frac{3}{4} + \frac{4}{4} = \frac{7}{4} \qquad \frac{4}{4} + \frac{3}{4} = \frac{7}{4}$$

$$\frac{2}{7} + \left(\frac{1}{7} + \frac{3}{7}\right) = \left(\frac{2}{7} + \frac{1}{7}\right) + \frac{3}{7}$$

4. Subtracting pairs of fractional numbers, then reversing the order of the pairs to see that subtraction is not commutative. For example:

$$\frac{3}{8} - \frac{1}{8} = \frac{2}{8} \qquad \frac{1}{8} - \frac{3}{8} \neq \frac{2}{8}$$

5. Subtracting using three fractional numbers, 2 numbers at a time; then changing the order of subtraction to see that subtraction is not associative. For example:

$$\left(\frac{8}{9} - \frac{5}{9}\right) - \frac{1}{9} = \frac{3}{9} - \frac{1}{9} = \frac{2}{9}$$

$$\frac{8}{9} - \left(\frac{5}{9} - \frac{1}{9}\right) = \frac{8}{9} - \frac{4}{9} = \frac{4}{9}$$

6. Using such examples as the following to discover the identity element of addition and subtraction of fractional numbers. For example:

$$\frac{2}{3} + 0 = 0 + \frac{2}{3} = \frac{2}{3}$$

$$\frac{4}{5} - 0 = \frac{4}{5}, \text{ but } 0 - \frac{4}{5} \neq \frac{4}{5}$$

PROPERTIES OF MULTIPLICATION AND DIVISION OF FRACTIONAL NUMBERS

Knowledge of properties of whole numbers may be used to determine whether these also hold true for multiplication and division of fractional numbers.

The Integers

1. NATURE OF INTEGERS

Situations involving gain and loss, rise and fall, deposit and withdrawal are familiar to most children. There should be much discussion to associate this practical experience with extension of the number system to include negative numbers. Activities like the following can be used to develop understanding of integers:

a. Weather reported as 10° below zero, may be written as $^{-}10°$ after a thermometer has been examined.

b. Time before rocket blast off may be counted down, the significance of zero discussed and the time written as $^{-}4$, $^{-}3$, $^{-}2$, $^{-}1$, 0.

c. An event occurring in 55 BC may be discussed, "zero" identified and the time written as $^{-}55$.

d. Ocean depth recorded as 2000 feet below sea level may be discussed, comparison made with a mountain peak of 3000 feet elevation, the "zero" identified and a diagram drawn. Distances may be written as $^{-}2000$ and $^{+}3000$.

e. The number line may be used to illustrate the route of a Scout troop that hikes 5 miles east, then 1 mile west, "zero" established as the starting point, and the distance recorded as $^{+}5$ and $^{-}1$.

After discussion of many activities of this type, pupils should be able to generalize that positive integers may be used to denote change to the right or upward, and that negative integers may be used to denote change to the left or downward. Care must be taken to insure that children understand that the symbols + and − are part of the numerals for integers and not operational symbols for addition and subtraction. If symbols for integers are written in the raised position and read as "positive" and "negative," much confusion can be avoided.

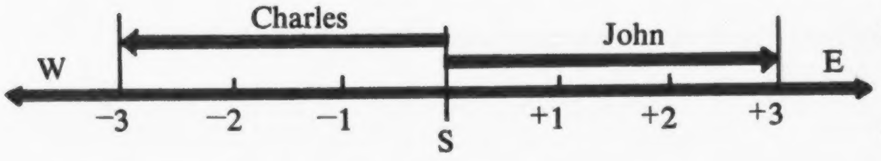
2. ORDER OF INTEGERS

As negative numbers are discussed, pupils may begin to use the number line extended to the left of zero. A thermometer represents the most familiar model and can provide a starting point for discussion. Children usually understand that 0 is colder or less warm than a reading of 10° above zero, that 3° below zero or $^{-}3°$ is colder or less warm than 2° above zero. From this beginning, the number line, both horizontal and vertical, can be used to answer such questions as the following:

Which are true?		Which are false?	
Is $^{+}6 > {}^{+}5$?	(T)	Is $^{-}7 < {}^{+}3$?	(T)
Is $^{+}5 < {}^{+}9$?	(T)	Is $0 < {}^{-}3$?	(F)
Is $^{-}3 < {}^{-}2$?	(T)	Is $^{-}4 = {}^{+}4$?	(F)
Is $^{-}4 < {}^{-}7$?	(F)	Is $0 < {}^{+}1$?	(T)

As relative values of positive and negative numbers are understood, the idea of opposites may be introduced through discussion of such problems as the following:

a. Starting from the same street corner, John walks 3 blocks east and Charles walks 3 blocks west. Does the number line below represent the situation?



Can $^{-}3$ and $^{+}3$ be thought of as opposites? Why?

b. Mary earns 25¢ and loses it on the playground. How much money does she then have? A gain of 25 and a loss of 25 is equal to zero, $^{+}25$ and $^{-}25 = 0$. May $^{+}25$ and $^{-}25$ be thought of as opposites?

When the relationship of opposites is understood, the generalization can be stated and the term additive inverse applied.

Use should be made of the number line in developing understanding of all operations on integers. If an integer is understood to be less than any integer to the right or above it, and greater than any to its left or below it, movement of concrete objects along the line should enable children to understand and express basic principles underlying algorithms in the four operations.

Using the number line, opportunity should be provided to show many such operations as the following:

Balance of 0. Deposit of $10. Withdrawal of $7.

Balance of $5. Deposit of $3. Withdrawal of $8.

Balance of $15. Deposit of $1. Withdrawal of $18.

Debt of $10. Deposit of $25. New balance of $15.

3. OPERATIONS ON INTEGERS

Addition: By demonstrating on the number line, children may complete such sentences as the following:

a. $^{+}6 + ^{+}3 = \square$

b. $^{+}2 + ^{+}8 = \square$

c. $^{-}3 + ^{-}5 = \square$

d. $^{-}8 + ^{-}5 = \square$

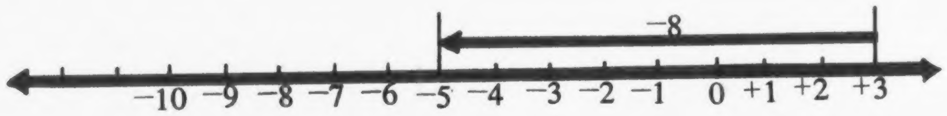
e. $^{+}5 + ^{-}3 = \square$

f. $^{+}3 + ^{-}8 = \square$

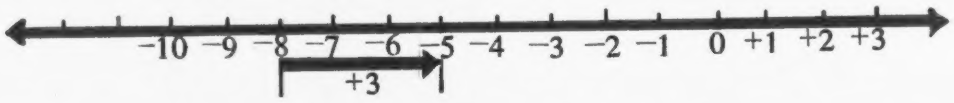
g. $^{-}8 + ^{+}3 = \square$

h. $^{-}2 + ^{+}6 = \square$

f.



g.



Does the commutative property hold for addition of integers?

Sums of integers where one is positive and one negative may also be introduced by leading children to discover a pattern which will enable them to fill in blanks remaining in each column below:

$4 + 1 = 5$
$4 + 0 = \_\_$
$4 + ^{-}1 = \_\_$
$4 + ^{-}2 = \_\_$
$4 + ^{-}3 = \_\_$
$4 + ^{-}4 = \_\_$
$4 + ^{-}5 = \_\_$

$3 + 1 = 4$
$3 + 0 = \_\_$
$3 + ^{-}1 = \_\_$
$3 + ^{-}2 = \_\_$
$3 + ^{-}3 = \_\_$
$3 + ^{-}4 = \_\_$
$3 + ^{-}5 = \_\_$

$2 + 1 = 3$
$2 + 0 = \_\_$
$2 + ^{-}1 = \_\_$
$2 + ^{-}2 = \_\_$
$2 + ^{-}3 = \_\_$
$2 + ^{-}4 = \_\_$
$2 + ^{-}5 = \_\_$

Sums of integers where both are negative may be introduced through discovery of a pattern which will enable them to fill in blanks remaining in each column below:

$5 + ^{-}3 = 2$
$4 + ^{-}3 = 1$
$3 + ^{-}3 = 0$
$2 + ^{-}3 = \_\_$
$1 + ^{-}3 = \_\_$
$0 + ^{-}3 = \_\_$
$^{-}1 + ^{-}3 = \_\_$
$^{-}2 + ^{-}3 = \_\_$
$^{-}3 + ^{-}3 = \_\_$

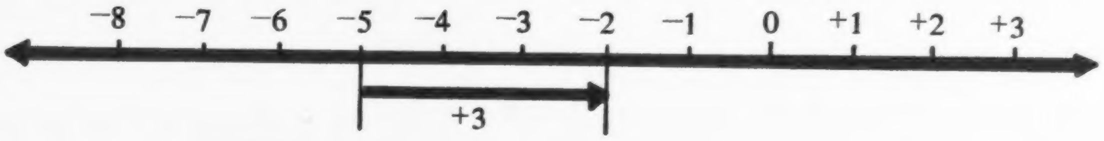
$3 + ^{-}4 = ^{-}1$
$2 + ^{-}4 = ^{-}2$
$1 + ^{-}4 = ^{-}3$
$0 + ^{-}4 = \_\_$
$^{-}1 + ^{-}4 = \_\_$
$^{-}2 + ^{-}4 = \_\_$
$^{-}3 + ^{-}4 = \_\_$
$^{-}4 + ^{-}4 = \_\_$
$^{-}5 + ^{-}4 = \_\_$

$3 + ^{-}5 = ^{-}2$
$2 + ^{-}5 = ^{-}3.$
$1 + ^{-}5 = ^{-}4$
$0 + ^{-}5 = \_\_$
$^{-}1 + ^{-}5 = \_\_$
$^{-}2 + ^{-}5 = \_\_$
$^{-}3 + ^{-}5 = \_\_$
$^{-}4 + ^{-}5 = \_\_$
$^{-}5 + ^{-}5 = \_\_$

After discussion of results obtained by use of number line and through observation of developing patterns, children should be able to state and use basic principles that apply to addition of positive and negative integers.

Subtraction: Before attempting subtraction involving negative numbers it is important that subtraction has been recognized as the inverse of addition, the process of finding a missing addend when the sum and one addend are known.

With the number sentence $^{-}2 - {}^{-}5 = \square$ rewritten as $^{-}5 + \square = {}^{-}2$ the number line may be used to find the missing addend. Starting at the point on the line named by the known addend, $^{-}5$, and moving to the point named by the sum, $^{-}2$, three units on the line are passed. The direction is to the right or positive so the missing addend is $^{+}3$.



Other number sentences similar to the following may be solved in a like manner:

$^{-}4 - {}^{+}2 = \square$ becomes $^{+}2 + \square = {}^{-}4$

$^{+}7 - {}^{-}4 = \square$ becomes $^{-}4 + \square = {}^{+}7$

$^{+}3 - {}^{+}5 = \square$ becomes $^{+}5 + \square = {}^{+}3$

Number sentences involving negative numbers may be solved in this way until children can reach generalizations stated on pages 69-70 of Rational Numbers.

Patterns may also be used to develop this understanding as children find numbers that will enable them to fill blanks in sequences like those below:

$^{+}7 - {}^{+}4 = {}^{+}3$
$^{+}6 - {}^{+}4 = {}^{+}2$
$^{+}5 - {}^{+}4 = {}^{+}1$
$^{+}4 - {}^{+}4 =$ __
$^{+}3 - {}^{+}4 =$ __
$^{+}2 - {}^{+}4 =$ __
$^{+}1 - {}^{+}4 =$ __
$0 - {}^{+}4 =$ __
$^{-}1 - {}^{+}4 =$ __
$^{-}2 - {}^{+}4 =$ __

$^{-}7 - {}^{+}4 = {}^{-}11$
$^{-}7 - {}^{+}3 = {}^{-}10$
$^{-}7 - {}^{+}2 = {}^{-}9$
$^{-}7 - {}^{+}1 = {}^{-}8$
$^{-}7 - 0 = {}^{-}7$
$^{-}7 - {}^{-}1 = {}^{-}6$
$^{-}7 - {}^{-}2 =$ __
$^{-}7 - {}^{-}3 =$ __
$^{-}7 - {}^{-}4 =$ __
$^{-}7 - {}^{-}5 =$ __
$^{-}7 - {}^{-}6 =$ __
$^{-}7 - {}^{-}7 =$ __
$^{-}7 - {}^{-}8 =$ __
$^{-}7 - {}^{-}9 =$ __

Multiplication: Understanding of multiplication as repeated addition may be used to introduce multiplication of integers. Examples like the following may be used:

$$3 \times 4 = 4 + 4 + 4 = 12$$

$$3 \times (^{-}4) = (^{-}4) + (^{-}4) + (^{-}4) = {}^{-}12$$

$$(^{-}3) \times 4 = 4 \times (^{-}3) \quad \text{Commutative property.}$$
$$= (^{-}3) + (^{-}3) + (^{-}3) + (^{-}3) = {}^{-}12$$

There should be much discussion and many examples like the one above before children are asked to state the generalization of the way a positive integer and a negative integer are multiplied.

The decreasing factor approach may provide the best introduction to the multiplication of two negative numbers. After using this approach and developing several patterns, children should be able to generalize that the product of two negative integers is a positive number.

Division: The idea that division is the inverse of multiplication may be used to introduce division of integers. Examples like the following may be used:

$$12 \div 3 = 4;\ \frac{12}{3} = 4 \quad \text{because} \quad 3 \times 4 = 12$$

Inverse operation

$$^{-}12 \div {}^{-}4 = 3;\ \frac{^{-}12}{^{-}4} = 3 \quad \text{because} \quad 3 \times (^{-}4) = {}^{-}12$$

$$^{-}12 \div {}^{-}3 = 4;\ \frac{^{-}12}{^{-}3} = 4 \quad \text{because} \quad (^{-}3) \times 4 = {}^{-}12$$

$$12 \div {}^{-}3 = {}^{-}4;\ \frac{12}{^{-}3} = {}^{-}4 \quad \text{because} \quad (^{-}3) \times (^{-}4) = 12$$

When there has been careful discussion and practice with examples like the above, children should be able to formulate generalizations for division based on the generalizations for multiplication.

Negatives of Fractional Numbers

1. THE NATURE OF NEGATIVE FRACTIONAL NUMBERS

Ideas related to the nature of fractional numbers and integers also apply to negative fractional numbers. Each fractional number has an opposite, a negative fractional number, an equal distance to the left of zero along the number line. The sum of pairs of opposite fractional numbers is always zero.

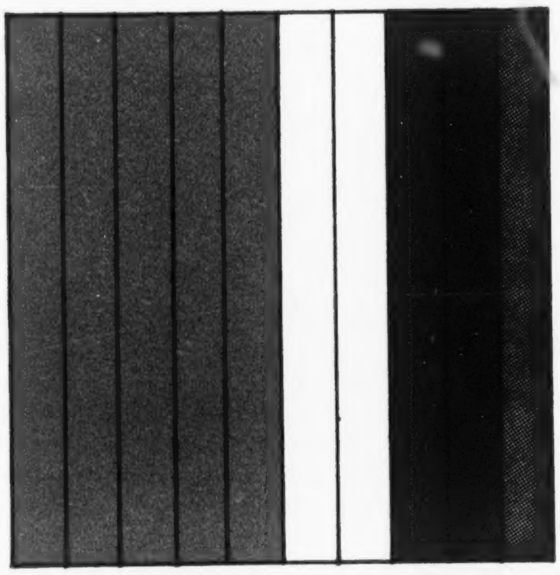
2. OPERATIONS ON POSITIVE AND NEGATIVE FRACTIONAL NUMBERS

Since operations on positive and negative fractional numbers are performed in the same way as operations using integers, there should be little difficulty in developing algorithms.

Decimals

1. THE NATURE OF DECIMALS

Decimals may be introduced through use of models of regions separated into tenths. As parts of the region are colored, children may name the fractional number identified with the shaded part.



The big region is one unit.		(1.0)
How many spaces are in the region?		(10)
What is another way to name the unit?	$\frac{10}{10}$	(1.0)
What part of the region is gray?	$\frac{1}{10}$	(.1)
What part of the region is lavender?	$\frac{5}{10}$	(.5)
What part of the region is lavender and white?	$\frac{7}{10}$	(.7)
What part of the region is lavender, white, blue , and gray?	$\frac{10}{10}$	(1.0)

The fact that tenths can be written using decimal notation may be introduced, the questions above used and answers written using decimals.

A model of the same size may be further subdivided to show ten rows of ten spaces each, similar questions asked and decimal notation for hundredths introduced. Transparencies with overlays to show tenths, hundredths, and thousandths may also be used to show relationships. Graph paper squared in tenths may be used by children to construct individual models.

If responses to such questions as the above are recorded on a chart, relationships may be observed and understanding of a decimal as a fractional number with some power of ten as a denominator may be further developed.

Tenths		Hundredths		Thousandths	
Fractions	Decimals	Fractions	Decimals	Fractions	Decimals
$\frac{1}{10}$	.1	$\frac{10}{100}$	.10	$\frac{100}{1000}$	.100
$\frac{7}{10}$	.7	$\frac{70}{100}$	.70	$\frac{700}{1000}$	.700
$1\frac{5}{10}$ or	1.5	$1\frac{50}{100}$ or	1.50	$1\frac{500}{1000}$ or	1.500
$\frac{15}{10}$		$\frac{150}{100}$		$\frac{1500}{1000}$	

Decimals may also be linked to children's knowledge of values of dollars, dimes, and pennies. An array of pennies, ten rows of ten each, or a chart showing this type of array, may be used to develop understanding of hundredths. Notation can be associated with what is already known about use of the decimal point in writing money: 16 pennies = \$.16; 8 pennies = \$.08.

Such activities as the following may provide additional help in understanding the nature of decimals and their relation to fractional numbers:

Construct place value charts to use in reading and writing decimals.

Arrange the following decimals in order, beginning with the least—.36, .3, .03, 9.5, 4.16, .001, . . .

Use ruler, meter stick, or steel tape marked in tenths to measure heighths of children.

Use an odometer, "handy" grocery counter, pedometer or other devices for measuring which register in tenths of units.

Terminating Decimals: As understanding of the nature of decimals develops, it should be evident that it is simple to express $\frac{7}{10}$ or $\frac{58}{100}$ as decimals. Such questions as the following may lead children to name fractional numbers as equivalent fractions having powers of ten as denominators and to express the resulting fractions as decimals:

(a) $\frac{1}{2} = \frac{n}{10}$; (b) $\frac{2}{5} = \frac{n}{10}$; (c) $\frac{1}{4} = —$;
(d) $\frac{7}{25} = —$; (e) $\frac{7}{8} = —$.

What is the value of n in examples (a) and (b) above?

Write the decimal equivalent of (a) and (b) above.

In (c), (d), and (e) above, what powers of ten can be used as denominators to make equivalent fractions?

Write the decimal notations for (c), (d), and (e) above. (.25, .28, .875)

When the decimal equivalent is not readily seen, it may be necessary to perform the indicated division:

$$\frac{7}{8} \text{ means } 7 \div 8 \text{ or } \begin{array}{r} .875 \\ 8\overline{)7.000} \end{array}$$

If in performing the division a remainder of zero is obtained, the decimal is said to terminate.

Repeating Decimals: After division of decimals is familiar to children, it can be shown by division that some fractional numbers can be expressed as decimals which do not terminate and the digits of quotients repeat, such as in the following examples:

$$\frac{1}{3} = 3\overline{)1.000} = 0.333\ldots$$

$$\frac{1}{7} = 7\overline{)1.000000} = .142857142857\ldots$$

Such quotients are known as repeating decimals.

From the above it should be evident that all fractional numbers may be expressed in decimal notation.

Charts such as the following may help to bring out the patterns:

Fraction	Decimal Equivalent	Repeating (R) Terminating (T)	Prime Factors of Denominators
$\frac{1}{2}$	.5	T	2
$\frac{1}{3}$	$.3\overline{3}$	R	3
$\frac{1}{4}$	.25	T	2
$\frac{1}{5}$	.2	T	5
$\frac{1}{6}$	$.166\overline{6}$	R	2, 3
•	•	•	•
•	•	•	•
•	•	•	•

2. OPERATIONS ON DECIMALS

Addition: Addition of decimals may be introduced by using what children know about addition of fractional numbers. Decimals may be written as fractions and added.

$$.7 + .2 = \frac{7}{10} + \frac{2}{10} = \frac{9}{10} = .9$$

$$.75 + .21 = \frac{75}{100} + \frac{21}{100} = \frac{96}{100} = .96$$

$$\begin{aligned} .16 + .2 &= \frac{16}{100} + \frac{2}{10} = \frac{16}{100} + \frac{20}{100} \\ &= \frac{36}{100} = .36 \end{aligned}$$

$$\begin{aligned} .5 + .36 + .009 &= \frac{5}{10} + \frac{36}{100} + \frac{9}{1000} \\ &= \frac{500}{1000} + \frac{360}{1000} + \frac{9}{1000} \\ &= \frac{869}{1000} = .869 \end{aligned}$$

Addition of decimals may also be introduced by using knowledge of place value notation developed through use of place value charts.

$$\begin{array}{r} .7 \\ +\ .2 \\ \hline .9 \end{array} \qquad \begin{array}{r} .75 \\ +\ .21 \\ \hline .96 \end{array} \qquad \begin{array}{r} .16 \\ +\ .2 \\ \hline .36 \end{array} \qquad \begin{array}{r} .5 \\ .36 \\ +\ .009 \\ \hline .869 \end{array}$$

Through practice in writing decimals for addition and subtraction using a place value chart, children should be aware of the significance of the decimal point in writing numerals for decimals.

Subtraction: Understanding of addition of decimals can be applied to subtraction, either by expressing the decimal as a fraction or by using knowledge of place value notation. Since decimals are base ten numerals and use the same place value pattern as whole numbers, children should expect the decimals to behave in the operations of addition and subtraction in a way similar to whole numbers.

Multiplication: Multiplication of decimals may be related to what children know about multiplication of fractional numbers.

$$.3 \times .2 = \frac{3}{10} \times \frac{2}{10} = \frac{3 \times 2}{10 \times 10} = \frac{6}{100} = .06$$

$$5 \times .7 = \frac{5}{1} \times \frac{7}{10} = \frac{35}{10} = 3\frac{5}{10} = 3.5$$

$$\begin{aligned} .7 \times .21 &= \frac{7}{10} \times \frac{21}{100} = \frac{7 \times 21}{10 \times 100} \\ &= \frac{147}{1000} = .147 \end{aligned}$$

$$\begin{aligned} 8 \times 2.4 &= \frac{8}{1} \times 2\frac{4}{10} = \frac{8}{1} \times \frac{24}{10} = \frac{192}{10} \\ &= 19\frac{2}{10} = 19.2 \end{aligned}$$

As many operations of this type are performed, children should be able to generalize that ones multiplied by tenths gives tenths, tenths multiplied by tenths gives hundredths, and tenths multiplied by hundredths gives thousandths.

The operation can then be performed using decimal notation, products can be compared and generalization applied. As operations with decimals involving ten thousandths and hundred thousandths are performed, generalizations may be extended to include the idea that hundredths multiplied by hundredths gives ten thousandths.

After solving many problems like the following using fractions and changing products to decimals, a pattern should emerge. Children should be able to see that to place the decimal point correctly in the product, the total number of digits to the right of the decimal in numerals of factors may be counted.

Examples

$$.3 \times 2 = \frac{3}{10} \times 2 = \frac{6}{10} = .6$$

$$.9 \times .3 = \frac{9}{10} \times \frac{3}{10} = \frac{27}{100} = .27$$

$$4 \times .16 = 4 \times \frac{16}{100} = \frac{64}{100} = .64$$

$$.29 \times .5 = \frac{29}{100} \times \frac{5}{10} = \frac{145}{1000} = .145$$

$$.396 \times 4 = \frac{396}{1000} \times 4 = \frac{1584}{1000} = 1.584$$

$$.15 \times .09 = \frac{15}{100} \times \frac{9}{100} = \frac{135}{10{,}000} = .0135$$

Pattern

$1 \times 1 = 1$	$3 \times 1 = 3$
$1 \times .1 = .1$	$3 \times .1 = 0.3$
$1 \times .01 = .01$	$3 \times .01 = 0.03$
$1 \times .001 = .001$	$3 \times .001 = 0.003$
and	and
$.1 \times 1 = .1$	$.4 \times 1 = .4$
$.1 \times .1 = .01$	$.4 \times .1 = .04$
$.1 \times .01 = .001$	$.4 \times .01 = .004$

Multiplication of decimals may also be approached through use of renaming, commutative and associative properties as illustrated below:

$.6 \times .21$

$= (6 \times .1) \times (21 \times .01)$ renaming

$= (6 \times 21) \times (.1 \times .01)$ commutative and associative properties

$= 126 \times .001 = .126$

When the principle is understood and accepted, many activities like the following may be introduced:

> Solve the following by using fractional notation and changing answer to a decimal:
>
> $.7 \times .8 =$ $.5 \times 8.6 =$
>
> $4.25 \times 1.2 =$ $3.02 \times 5.8 =$

> Place the decimal point in products of the following:
>
> $.9 \times 3.2 = 2\ 8\ 8$
>
> $.4 \times .36 = 1\ 4\ 4$
>
> $.5 \times 1.82 = 9\ 1\ 0$
>
> $.05 \times 1.82 = 0\ 0\ 9\ 1\ 0$

Division: The division of decimals may be introduced by recalling what children know about the division of fractional numbers, by using understanding of division as the inverse of multiplication, and by employing knowledge of place value notation.

Fractional number approach:

$$.3 \div .9 = \frac{3}{10} \div \frac{9}{10} = \frac{3}{10} \times \frac{10}{9} = \frac{30}{90} = \frac{3}{9} = \frac{1}{3}$$

Inverse of multiplication approach:

$.21 \div .7 = n$ $n \times .7 = .21$ $.3 \times .7 = .21$

$n = .3$

Place value approach:

Through multiplying by one the divisor is changed to a whole number and the algorithm is performed using the place value notation.

$$8.5\overline{)76.5} \quad \frac{76.5}{8.5} \times \frac{10}{10} = \frac{765}{85} \quad \begin{array}{r} 9 \\ 85\overline{)765} \end{array}$$

$$.85\overline{).765} \quad \frac{.765}{.85} \times \frac{100}{100} = \frac{76.5}{85} \quad \begin{array}{r} .9 \\ 85\overline{)76.5} \end{array}$$

$$.085\overline{)7.65} \quad \frac{7.65}{.085} \times \frac{1000}{1000} = \frac{7650}{85} \quad \begin{array}{r} 90 \\ 85\overline{)7650} \end{array}$$

Patterns which can be seen by using the different forms of division may be helpful:

$39 \div 1 = 39.$	$\frac{39}{1} = 39.$	$\begin{array}{r} 39. \\ 1\overline{)39} \end{array}$
$39 \div 10 = 3.9$	$\frac{39}{10} = 3.9$	$\begin{array}{r} 3.9 \\ 10\overline{)39.} \end{array}$
$39 \div 100 = 0.39$	$\frac{39}{100} = 0.39$	$\begin{array}{r} .39 \\ 100\overline{)39.} \end{array}$
$39 \div 1000 = 0.039$		

$48 \div 3 = 16$

$4.8 \div 3 = 1.6$

$.48 \div 3 = 0.16$

$.048 \div 3 = 0.016$

After much practice children should be able to state the generalization that:

> When the divisor is a whole number, the decimal point will have the same position in the quotient as it has in the dividend.

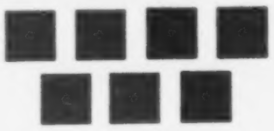
As a check, the quotient should be multiplied by the divisor to determine if the product equals the dividend.

The habit of estimating answers before solving problems is particularly important in checking operations using decimals. Estimating quotients before solving division examples makes the above generalization more meaningful. Estimations should be recorded for every example.

Ratio and Proportion

1. THE NATURE OF RATIO AND PROPORTION

The idea that a ratio is a comparison of two numbers by division may be developed through use of concrete objects which can be compared. For example: red blocks and blue blocks, balls and bats, chairs and tables, books and children. Illustrations such as the following may be used and the answers expressed first as ordered pairs, then as fractional numbers, indicated division and ratios:

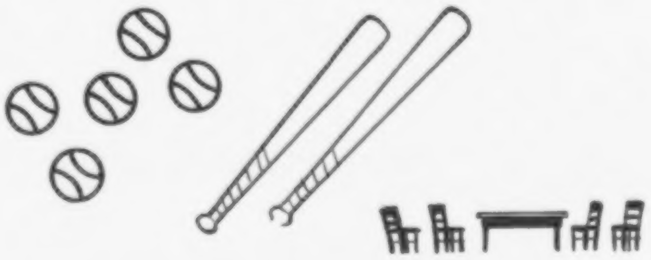


a. What is the ratio of red blocks to blue blocks?

(4, 3) $\frac{4}{3}$ $4 \div 3$ 4:3

b. What is the ratio of blue blocks to red blocks?

(3, 4) $\frac{3}{4}$ $3 \div 4$ 3:4



c. What is the ratio of balls to bats?

(5, 2) $\frac{5}{2}$ $5 \div 2$ 5:2

d. What is the ratio of table to chairs?

(1, 4) $\frac{1}{4}$ $1 \div 4$ 1:4

Story problems such as the following may be used in developing understanding of ratio:

a. There are 30 cookies for ten children. What is the ratio of cookies to children?

(30, 10) $\frac{30}{10}$ $30 \div 10$ 3:1

Ratios are usually expressed in lowest terms.

b. Three adults went on a camping trip with twenty scouts. What was the ratio of scouts to adults?

(20, 3) $\frac{20}{3}$ $20 \div 3$ 20:3

c. If two candy bars cost 12 cents, what is the ratio of cost to candy bars?

(12, 2) $\frac{12}{2}$ $12 \div 2$ 6:1

Proportion: Understanding of a porportion as a statement that two or more ratios are equal may be related to knowledge of equivalent fractions:

$$\frac{1}{2} = \frac{2}{4} = \frac{4}{8} \ldots$$

Such questions as the following may be used:

Is the ratio of 2 candy bars for 12¢ the same as 5 bars for 30¢?

Is the ratio of 3 adults to 20 scouts the same as 9 adults to 60 scouts?

Is the ratio of 30 cookies to 10 children the same as 60 cookies for 15 children?

2. PROPERTIES AND OPERATIONS

Since a ratio is a rational number, operations and properties for rational numbers apply to proportions.

Per Cent

1. NATURE OF PER CENT

The fact that a per cent is a special kind of ratio and that the term per cent means per centum or per one hundred should be discussed. If children are familiar with ratios written in the form 3:5 and read as 3 to 5 or 3 per 5, then 7% can be read as 7 per 100 and written in the following forms:

7%; 7:100; $\frac{7}{100}$; .07; 7 per 100

(Using the idea of sets 7% means that for every 7 in one set there are 100 in the second set.)

Problems involving per cent may be solved through the use of proportions.

$$\frac{3}{5} = \frac{n}{100}$$

$$\frac{n}{5} = \frac{60}{100}$$

$$\frac{3}{n} = \frac{60}{100}$$

Properties for Four Arithmetic Operations for Rational Numbers

Children may test the properties—closure, commutativity, associativity, identity, inverse, and distributivity (under two operations)—to see if they hold true for operations on numbers from various sets. For example:

Addition—Using the set of even numbers:

1. $2 + 6 = 8$, an even number	Closure holds $a + b = c$
2. $2 + 4 = 4 + 2;\ 6 = 6$	Commutativity holds $a + b = b + a$
3. $4 + (8 + 6) = (4 + 8) + 6;\ 4 + 14 = 12 + 6;\ 18 = 18$	Associativity holds $a + (b + c) = (a + b) + c$
4. $6 + 0 = 0 + 6 = 6$	Identity element (0) $a + 0 = 0 + a = a$
5. No inverse element	Additive inverse $a + {}^{-}a = 0$

Subtraction—Using the set of fractional numbers:

1. $\frac{5}{6} - \frac{1}{6} = \frac{4}{6}$, a fractional number $\frac{1}{6} - \frac{5}{6}$ is not a fractional number	Closure does not hold
2. $\frac{5}{6} - \frac{1}{6} \neq \frac{1}{6} - \frac{5}{6}$	Commutativity does not hold
3. $\left(\frac{5}{7} - \frac{1}{7}\right) - \frac{2}{7} \neq \frac{5}{7} - \left(\frac{1}{7} - \frac{2}{7}\right)$	Associativity does not hold
4. $\frac{5}{9} - 0 = \frac{5}{9} \qquad 0 - \frac{5}{9} \neq \frac{5}{9}$	Identity element (0) does not hold on the left but does on the right

Multiplication—Using the set of decimals:

1. $.3 \times .5 = .15$, a decimal	Closure holds
2. $.05 \times .6 = .6 \times .05 \qquad .030 = .030$	Commutativity holds
3. $(.3 \times .2) \times .7 = .3 \times (.2 \times .7)$ $.06 \times .7 = .3 \times .14 \qquad .042 = .042$	Associativity holds
4. $.16 \times 1 = 1 \times .16 = .16$	Identity element (1) holds
5. $.16 \times 0 = 0 \qquad 0 \times .16 = 0$	Multiplication property of 0 holds
6. $.7 \times \frac{1}{.7} = \frac{.7}{.7} = 1$	Multiplicative inverse holds
Multiplication and addition	
7. $.6 \times (.3 + .02) = (.6 \times .3) + (.6 \times .02)$	Distributivity of multiplication over addition holds

Division—Using the set of fractional numbers:

1. $\frac{5}{6} \div \frac{5}{12} = \frac{5}{6} \times \frac{12}{5} = \frac{60}{30} = \frac{2}{1}$ Closure holds

2. $\frac{5}{8} \div \frac{1}{9} \neq \frac{1}{9} \div \frac{5}{8}$

   $\frac{5}{8} \times \frac{9}{1} \neq \frac{1}{9} \times \frac{8}{5}$ Commutativity does not hold

   $\frac{45}{8} \neq \frac{8}{45}$

3. $\frac{2}{3} \div \left(\frac{1}{4} \div \frac{1}{2}\right) \neq \left(\frac{2}{3} \div \frac{1}{4}\right) \div \frac{1}{2}$

   $\frac{2}{3} \div \frac{2}{4} \neq \frac{8}{3} \div \frac{1}{2}$ Associativity does not hold

4. $\frac{3}{5} \div 1 = \frac{3}{5}$

   $1 \div \frac{3}{5} = \frac{5}{3}$ Identity on right only

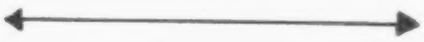
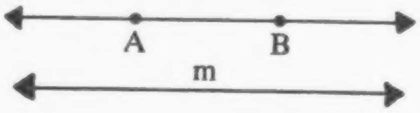
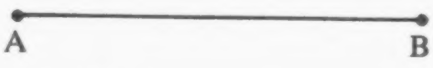
GEOMETRY

Geometry is the study of ideas of points, lines, curves, planes, and space. In the elementary school the approach is informal and intuitive. Relationships and properties are discovered and discussed before formal vocabulary is introduced. The language of geometry is the language of sets; that is, sets of points.

In the natural world geometric forms are everywhere observable. Man employs geometric shapes in the buildings which he designs, the utensils and instruments which he uses, the art with which he decorates his home, his clothes and his surroundings. It is desirable that children learn to see and identify some of these geometric designs, especially those that give a feeling for beauty and symmetry.

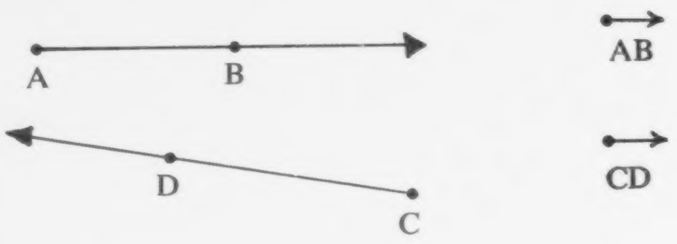
The early study of geometry must not be entirely incidental. Though it is hoped that teachers will capitalize upon opportunities to emphasize geometric ideas, there must be sequential development to reinforce concepts already discovered and to introduce new ones which follow logically. Thus, children will proceed at varying rates depending upon previous experience and level of maturity. Teachers should use many activities with concrete models which lead to better understanding of space figures and their relationships. Pupils who have had experiences manipulating models of closed surfaces such as balls and blocks will have greater readiness for understanding the connection between concrete objects and geometric ideas.

While geometry and measurement are treated as separate ideas in this guide, they are interrelated in classroom teaching. A study of line segments provides the necessary geometric background for studying linear measures. Plane regions are studied and then area of plane regions. Space regions are studied then volume or capacity of space regions. Thus size, the attachment of numbers to geometric objects, is interrelated with shape and space.

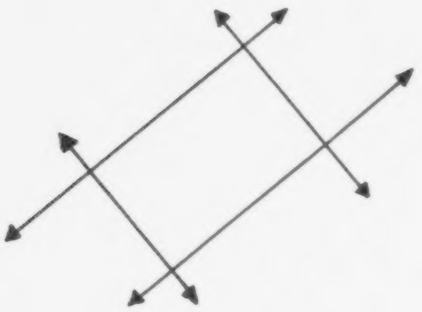
Mathematical Ideas	Illustrations and Explanations
Basic Ideas of Geometry The basic ideas of geometry are concepts of point, line, plane, curve, and simple closed curve. Physical models from the world are used to develop ideas.	
Point: A point can be thought of as an exact location in space which has no size or shape. Usually, a point is represented by a dot which in itself covers an infinite number of points.	Representation of a point .A .C .B Capital letters are used for labeling points.
Line: A line may be thought of as a set of points which extends endlessly in opposite directions. Unless otherwise stated, a line is thought to be straight but is not thought to have either breadth or thickness.	Representation of a line  Lines are named using two points of the line and the symbol $\longleftrightarrow$ over the names of the points or by using a lower case letter.  $\overleftrightarrow{AB}$ line m
Line segment: A line segment is a portion of a line which consists of two endpoints and all the points between them.	The endpoints are named by capital letters and the line segment is designated by $\overline{AB}$ and read "line segment AB."  $\overline{AB}$

Ray: A ray consists of one endpoint and all of the points of a line on one side of the endpoint.

Ray is written $\overrightarrow{AB}$ and read "ray AB." The letter naming the endpoint of the ray is always written first and the arrow indicating endless extension is placed over the letter naming the second point.

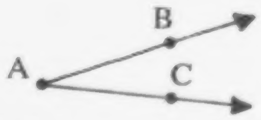


Plane: A plane may be thought of as a set of points forming a flat surface which extends in all directions without end.



Angle: An angle is a geometric figure consisting of two rays having the same endpoint.

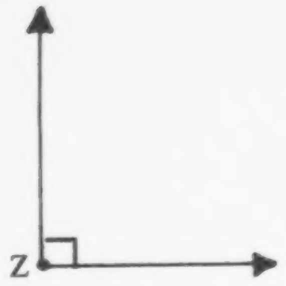
The symbol (∠) represents an angle.



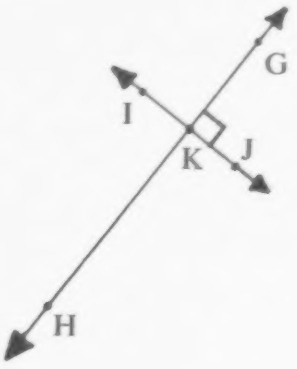
Point A is the common endpoint of the two rays and is called the *vertex* of the angle.

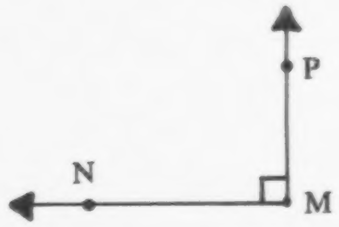
$\overrightarrow{AB} \cup \overrightarrow{AC}$ forms ∠BAC. (∠CAB) An angle can be named with the three letters BAC but the vertex letter must be in the center. The angle can be named by the single capital letter that names the vertex ∠A. The angle refers only to the set of points comprising the two rays and does not refer to any points off the rays.

Right angle: A right angle is the angle formed by two rays which form a square corner.



∠Z is a right angle

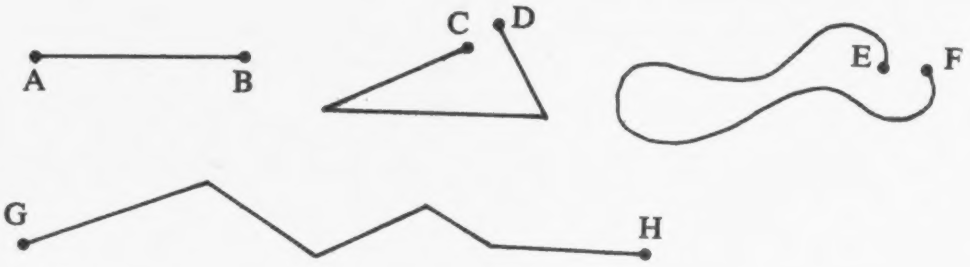




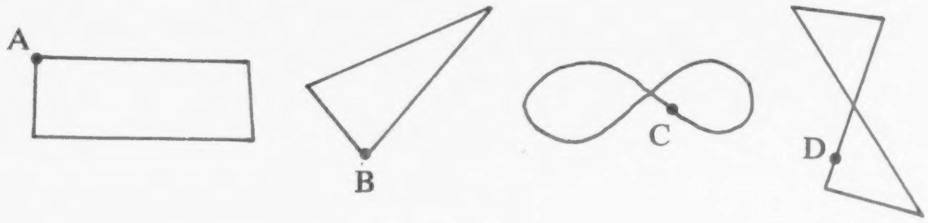
$\overrightarrow{MP} \cup \overrightarrow{MN}$ forms right angle PMN.

Lines HG and IJ form four right angles at K: ∠JKG, ∠JKH, ∠GKI, and ∠HKI.

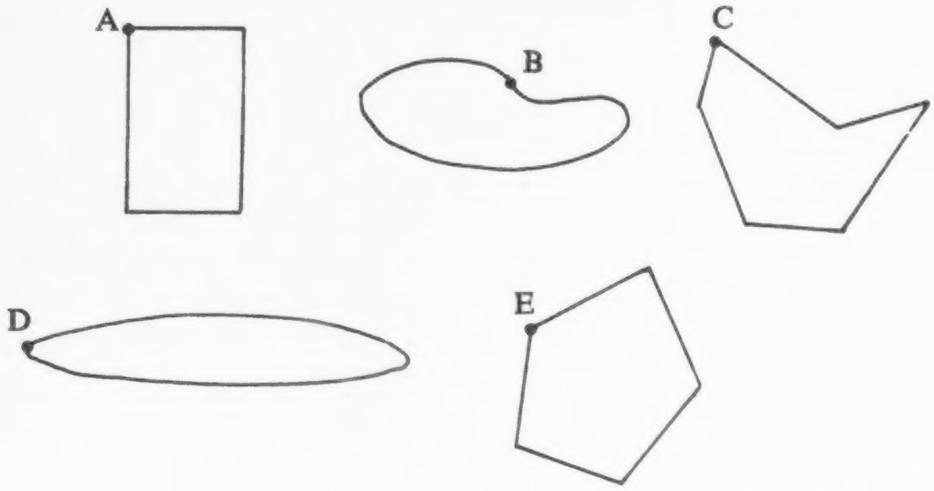
Path or curve: A path or curve is the set of all points passed through in moving from one point to another point in a plane.



Closed path or closed curve: A closed path or closed curve is a path which begins at one point and ends at the same point.



Simple closed curve: A simple closed curve is a closed curve that does not cross itself.

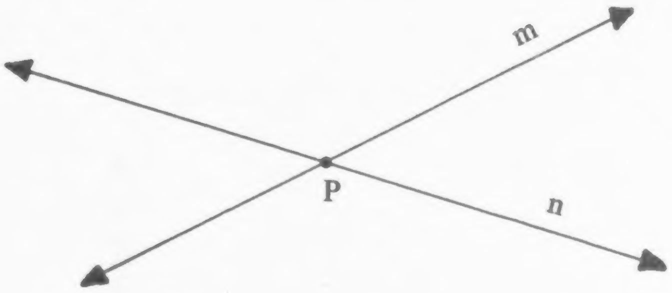


Relationships

1. INTERSECTION

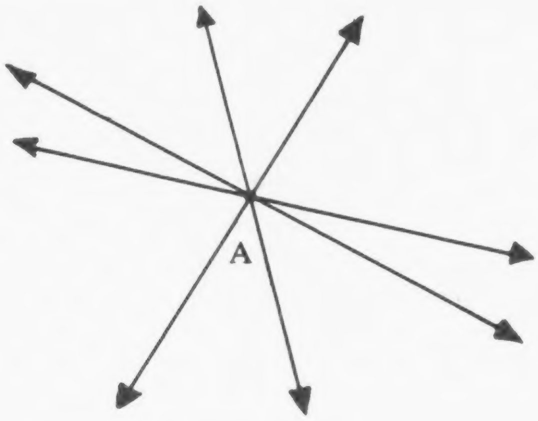
The intersection of two sets of points is the set of points common to the two sets.

Intersecting lines: When the sets of points which form two lines have only one point in common, the lines are said to intersect in this common point—the *point of intersection.* The lines are called *intersecting lines.*

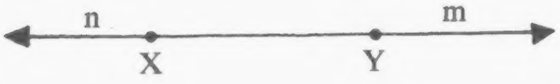


Line n and line m are *intersecting lines.* Point P is the point of intersection.

$n \cap m = P$

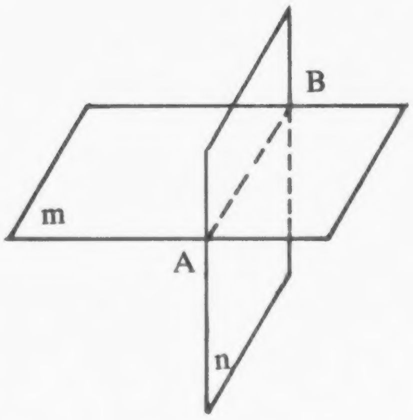


Through any point, A, an infinite number of lines may be drawn.



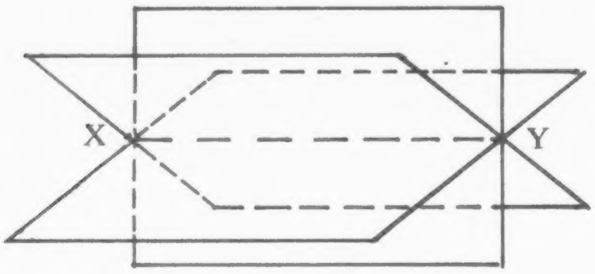
If 2 points of line n are also members of line m, then line m and line n are the same line. Any two points may be contained in one and only one line.

Intersecting planes: When the sets of points which form two planes have a set of points in common, the planes are said to intersect in this set of points. The set of points is a line. It is called the *line of intersection.* The planes are called *intersecting planes.*

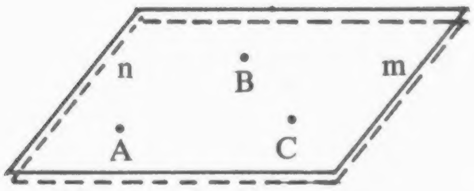


Plane m and plane n are intersecting planes. Line AB is the line of intersection.

$m \cap n = \overleftrightarrow{AB}$



An infinite number of planes can intersect in the line XY.

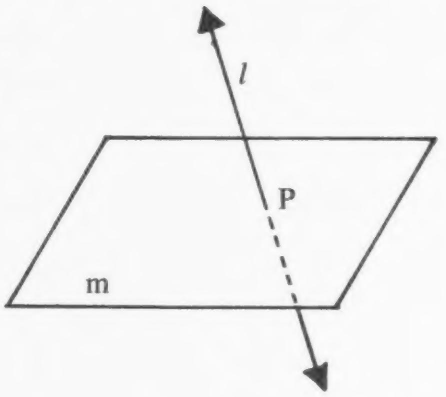


If three points not in the same line are in plane n and also in plane m, plane n and plane m are the same.

Any three points not in the same line determine one and only one plane.

Mathematical Ideas **Illustrations and Explanations**

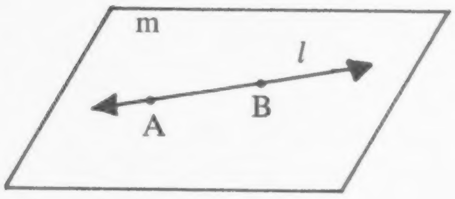
Line intersecting a plane: When the set of points which forms a line has only one point in common with the set of points which forms a plane, the line is said to intersect the plane in this point —the *point of intersection.*



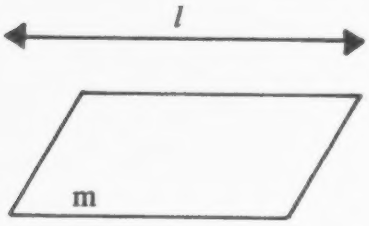
Line l intersects plane m in the point P.

$l \cap m = P$

If two points of line l are in plane m, then line l lies in plane m.



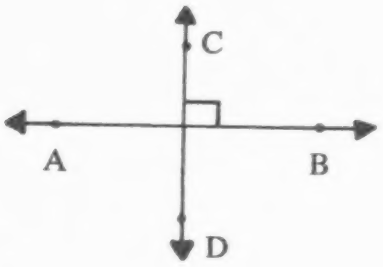
If a line joins two points in a plane, the line lies wholly in the plane. The intersection of the plane and the line is the line.



If a line and a plane have no point in common, the intersection of the two sets of points is the empty set. The line is parallel to the plane.

2. PERPENDICULARITY

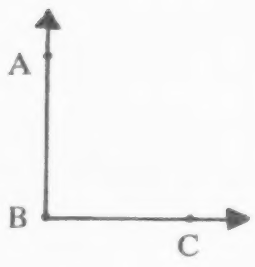
Perpendicular lines: If two lines intersect to form square corners or right angles they are called perpendicular lines. The symbol for "is perpendicular to" is " $\perp$ ".



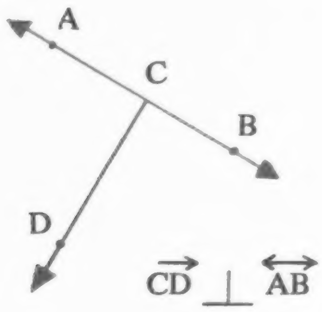
$\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$

or

$\overleftrightarrow{CD} \perp \overleftrightarrow{AB}$

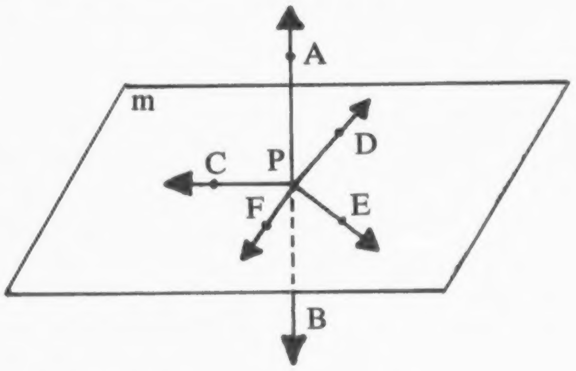


$\overrightarrow{BA} \perp \overrightarrow{BC}$



$\overrightarrow{CD} \perp \overleftrightarrow{AB}$

Line perpendicular to a plane: A line is perpendicular to a plane if it is perpendicular to every line in the plane passing through its point of intersection with the plane.



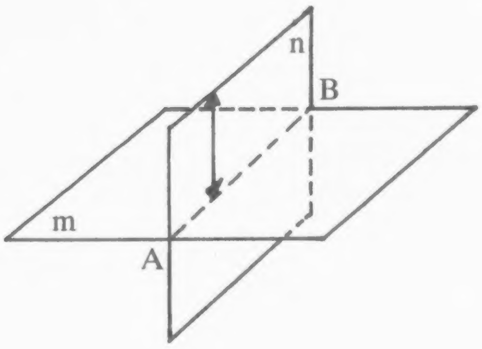
P is the point of intersection of line and the plane.

$\overleftrightarrow{AB} \perp \overleftrightarrow{EP}$, $\overleftrightarrow{AB} \perp \overleftrightarrow{PD}$,

$\overleftrightarrow{AB} \perp \overleftrightarrow{PC}$, $\overleftrightarrow{AB} \perp \overleftrightarrow{PF}$,

$\overleftrightarrow{AB} \perp$ plane m

Perpendicular planes: Two planes are perpendicular if a line in one plane is perpendicular to the second plane.

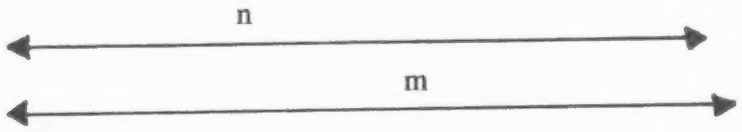


Plane n $\perp$ plane m

3. PARALLELISM

Parallel lines: Lines which lie in the same plane but have no points in common, do not intersect, are called parallel lines.

Rays and line segments are parallel if they are subsets of parallel lines.

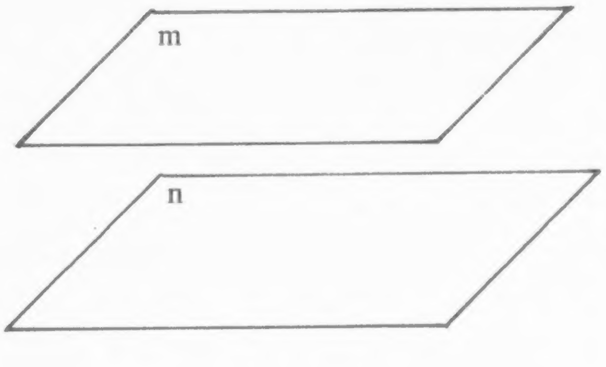


n || m line n is parallel to line m

Two lines in the same plane either intersect in one point or they are parallel.

Two lines not in the same plane are called *skew* lines.

Parallel planes: Two planes which never intersect are called parallel planes.



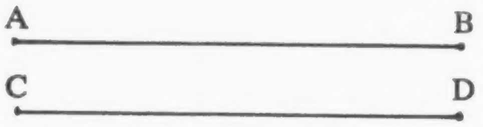
Plane m || plane n

Mathematical Ideas | **Illustrations and Explanations**

4. CONGRUENCY

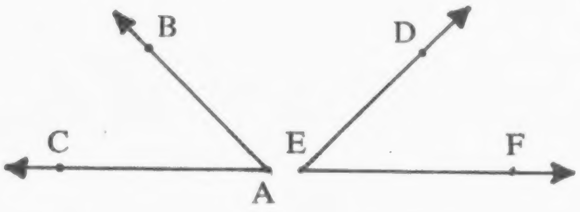
Congruency is the relationship of two geometric figures which have the same size and shape. The symbol for "is congruent to" is "≅".

Line segments: When two line segments have the same measure they are congruent.



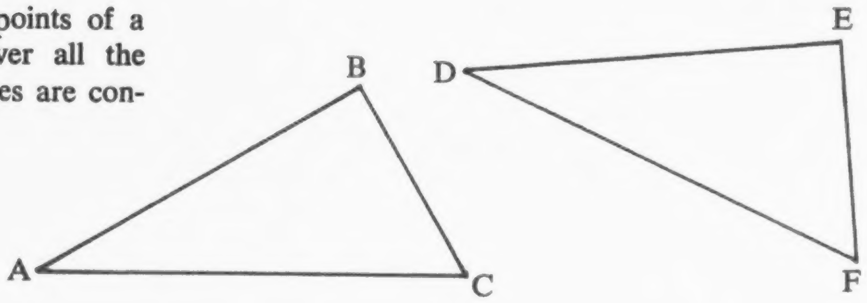
$\overline{AB} \cong \overline{CD}$

Angles: If all the points of one angle fit exactly over all the points of another angle, the angles are congruent.

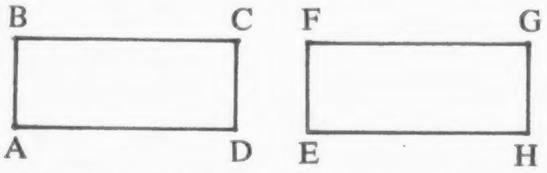


$\angle BAC \cong \angle DEF$

Plane Figures: If all the points of a plane figure fit exactly over all the points of another, the figures are congruent.



$\triangle ABC \cong \triangle DEF$



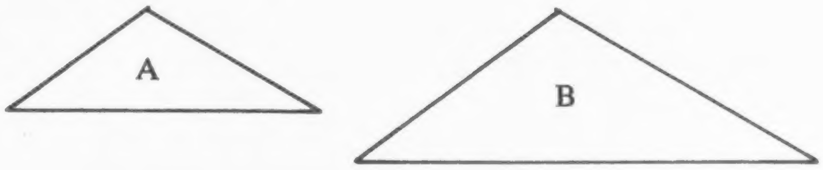
▭ ABCD ≅ ▭ EFGH

The two rectangles are congruent.

5. SIMILARITY

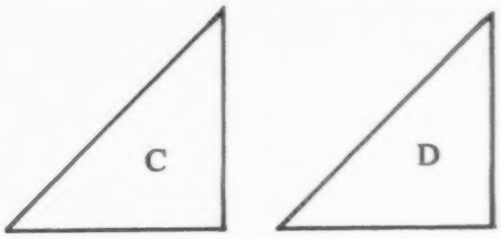
Geometric shapes or forms are similar, or similarity exists, when they have the same shape but not necessarily the same size. The symbol for "is similar to" is "∼".

All squares are similar.
All circles are similar.

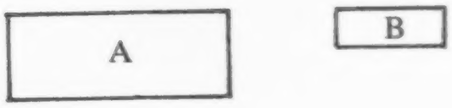


Triangles A and B are similar

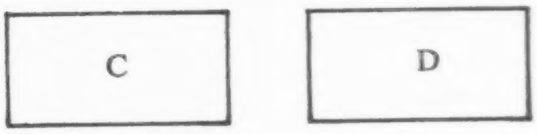
Scale drawing is based on similarity



Triangles C and D are similar and congruent.



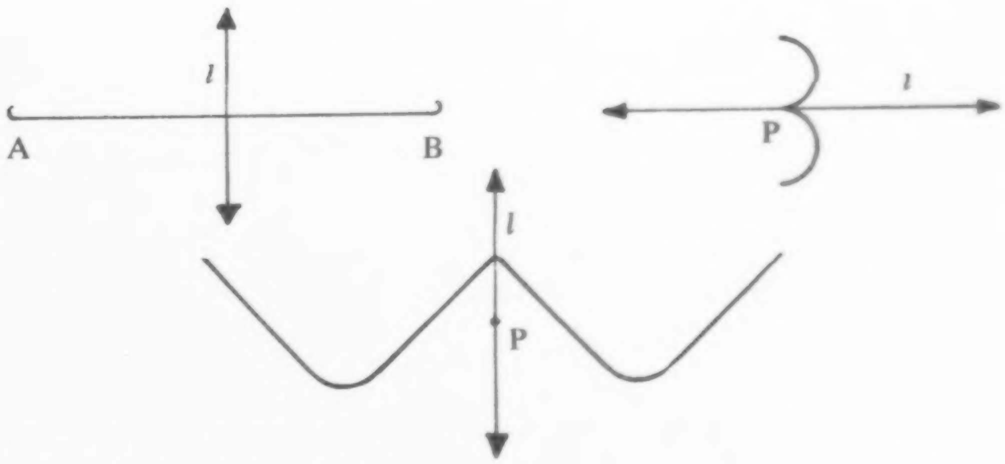
Rectangles A and B are similar, but not congruent.



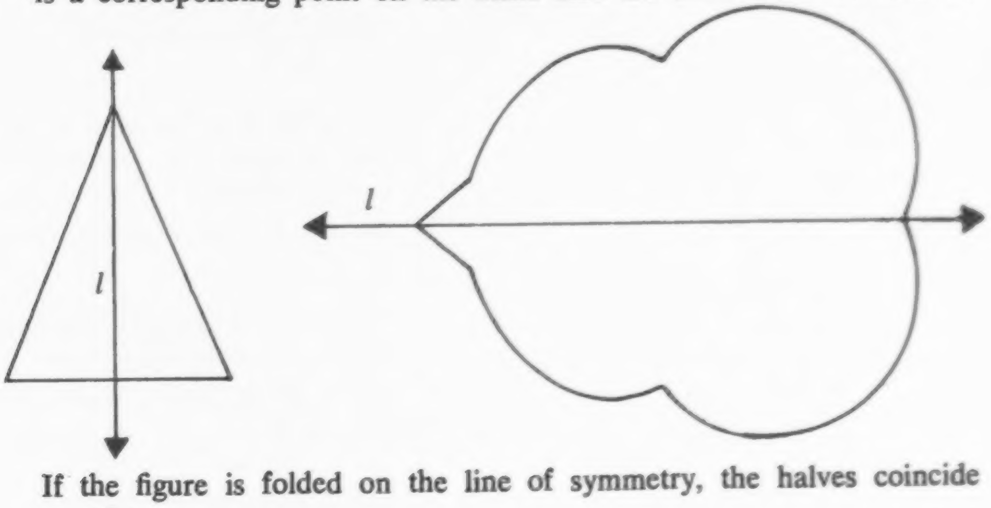
Rectangles C and D are similar and congruent.

6. SYMMETRY

Line symmetry: When a line can be drawn which divides a plane figure so that one side is an exact mirror image of the other, the figure is said to be symmetrical. The line is called the *line of symmetry* or the *axis of symmetry*.



Line l is the line of symmetry. For every point on one side of P there is a corresponding point on the other side the same distance from P.



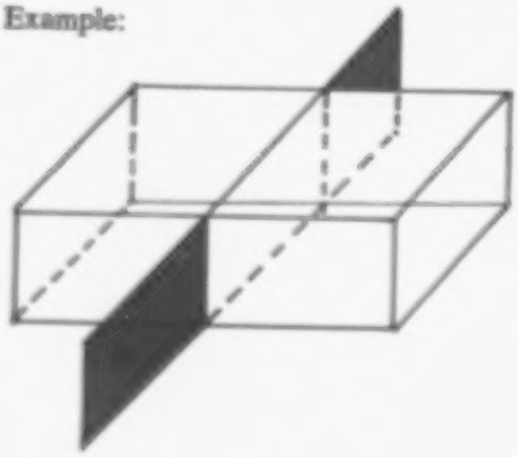
If the figure is folded on the line of symmetry, the halves coincide exactly.

For each point on one side of the line of symmetry there is a corresponding point on the other side.

Mathematical Ideas | Illustrations and Explanations

Plane symmetry: Some figures are symmetrical. When this is true a plane—the plane of symmetry—separates the figure so that one side is the exact mirror image of the other.

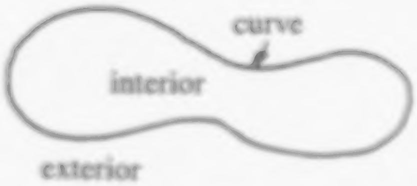
Example:

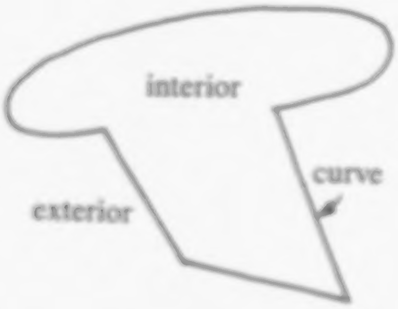


This rectangular prism is symmetrical with respect to plane m. It has three planes of symmetry.

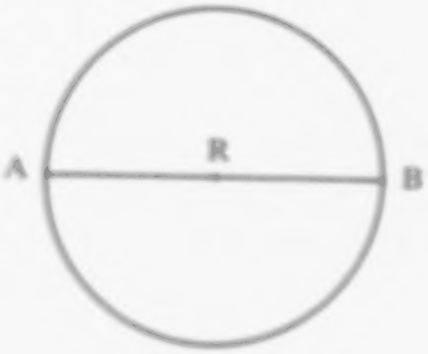
Plane Figures—Closed Paths

A simple closed curve separates a plane into three sets of points—the set of points *on* the curve, the set of points on the *interior* and the set of points on the *exterior*.





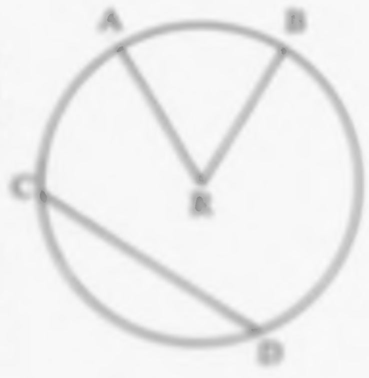
CIRCLE: A circle is a simple closed curve in a plane every point of which is the same distance from a given point in the interior called the center. The symbol for circle is ⊙ and is designated by the letter labeling the center.



A circle separates a plane into three sets of points—interior, the curve, and exterior sets of points.

$\overline{AB}$ is the diameter of ⊙ R. A *diameter* of a circle is a line segment which passes through the center point with its end points on the circle. Every circle has an infinite number of diameters.

$\overline{RB}$ and $\overline{RA}$ are radii of ⊙ R.



A *radius* of a circle is a line segment with one endpoint on the center of the circle and the other endpoint on the curve. Every circle has an infinite number of *radii*.

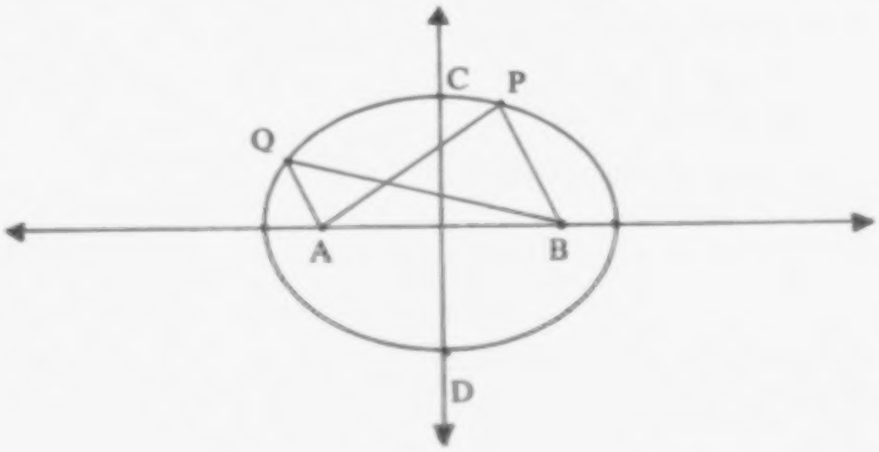
An *arc* of a circle is that part of the circle travelled in moving from one designated point to another designated point. $\overset{\frown}{AB}$, $\overset{\frown}{CD}$, $\overset{\frown}{BD}$, $\overset{\frown}{AC}$.

$\overline{CD}$ is a chord of ⊙ R.

A chord of a circle is the line segment with endpoints on the curve. The diameter is a special chord.

Mathematical Ideas	Illustrations and Explanations

Ellipse: An ellipse is a simple closed curve having two fixed points on the interior called the foci. Each point of the curve has associated with it a number which is the same for every point. This number is the sum of the measures of the distances from the point to the two fixed points.

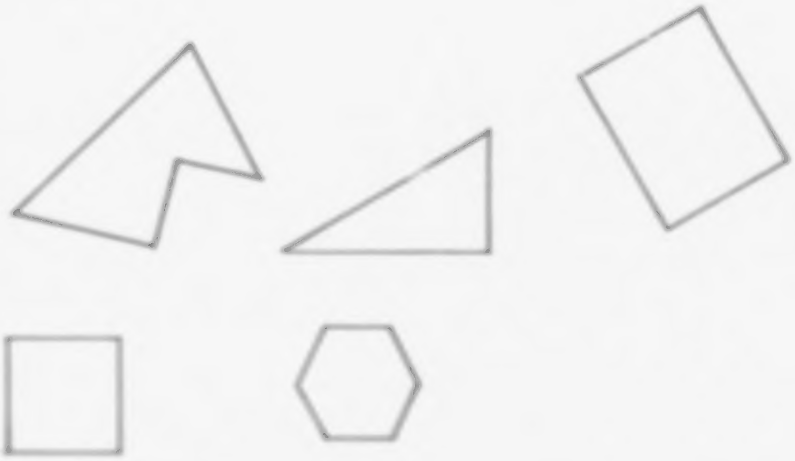


An ellipse has two lines of symmetry.

Points A and B are foci.

Polygon: A polygon is a simple closed curve formed by the union of three or more line segments. Two line segments with a common endpoint form an angle since line segments can be thought of as parts of rays. The common endpoints are called vertices of the polygon.

Examples:



A regular polygon has line segments or sides of the same measure and its angles are congruent.

Triangle: A triangle is a polygon with exactly three sides. The sides of a triangle may have one of three relationships:

three sides equal in measure (equilateral)

two sides equal in measure (isosceles)

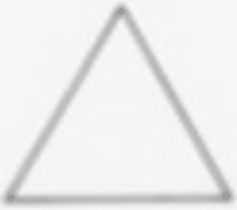
all three sides of different measures (scalene).

The angles of a triangle may have certain relationships:

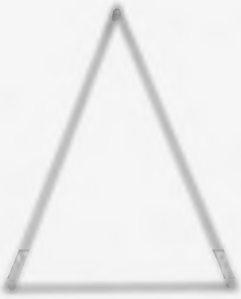
three angles that are congruent (equiangular)

two angles that are congruent (isosceles)

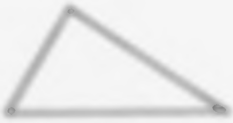
all three angles that are not congruent (acute)



equilateral
equiangular



isosceles



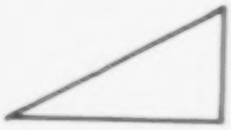
scalene
acute

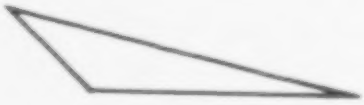
Mathematical Ideas **Illustrations and Explanations**

one angle a right angle (right)

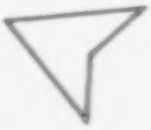
one angle greater than a right angle (obtuse)

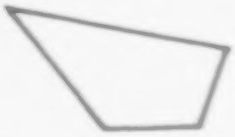
all angles less than a right angle (acute).


right

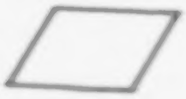

obtuse

Quadrilateral: A quadrilateral is a polygon with exactly four sides.

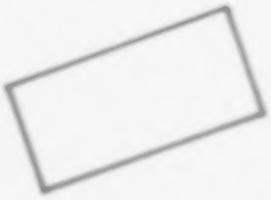




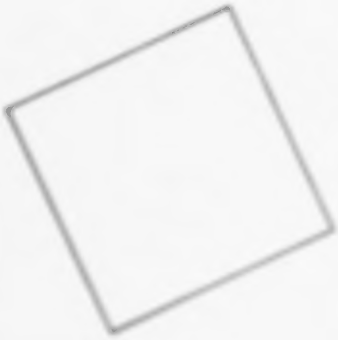
PARALLELOGRAM: A parallelogram is a quadrilateral having opposite sides parallel and equal in measure.



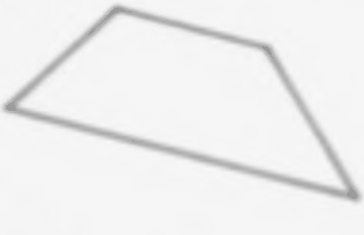
RECTANGLE: A rectangle is a quadrilateral having opposite sides parallel, equal in measure, and containing one right angle. A rectangle is a parallelogram.



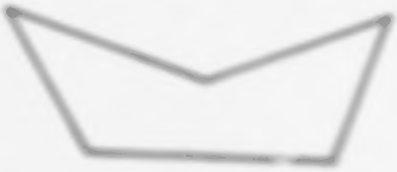
SQUARE: A square is a quadrilateral which is a rectangle having all sides of the same measure. A square is a rectangle and a parallelogram.

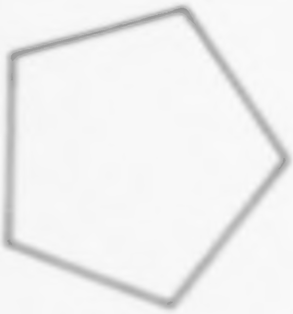


TRAPEZOID: A trapezoid is a quadrilateral having two and only two sides parallel.



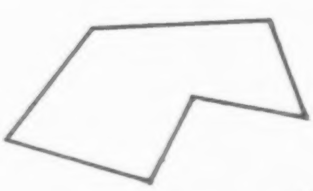
Pentagon: A pentagon is a polygon having exactly five sides.

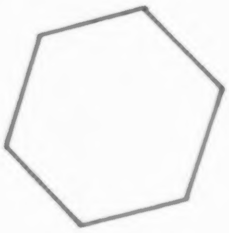




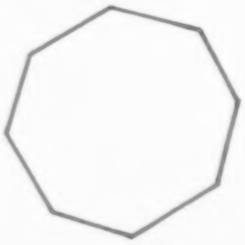
Mathematical Ideas	Illustrations and Explanations

Hexagon: A hexagon is a polygon having exactly six sides.



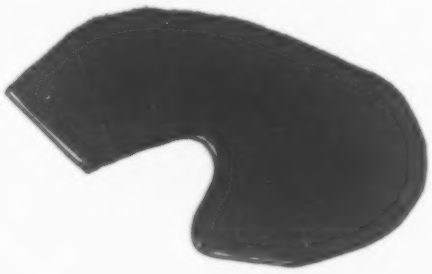


Octagon: An octagon is a polygon having exactly eight sides.



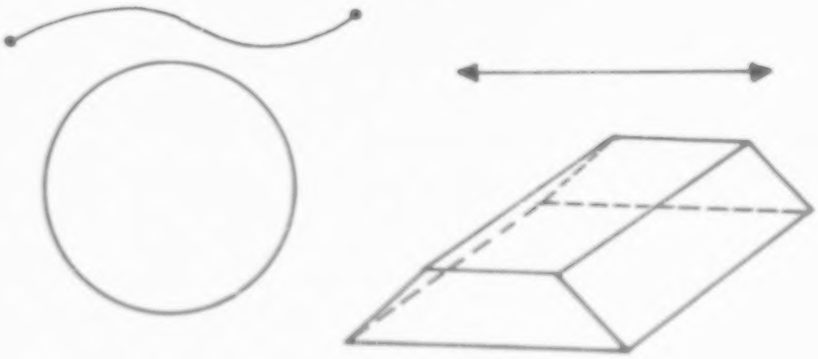
Plane Region

A plane region is the union of the curve and its interior set of points.

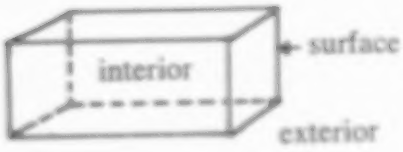


Space Figures—Closed Surfaces

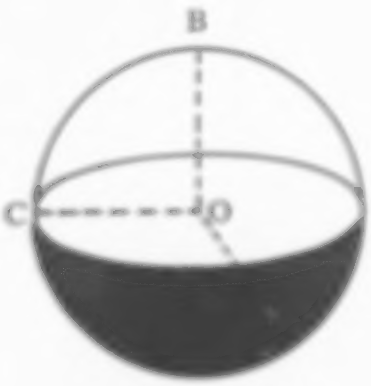
Any set of points is a *space figure*. Geometry includes the sets of points in all space as well as sets of points in a plane.



A closed surface separates space into three sets of points—those in the *interior*, those on the *surface*, and those on the *exterior*. A point in space lies either on the surface, in the interior set of points, or on the exterior set of points.



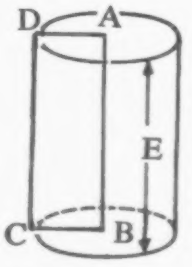
Sphere: A sphere is a closed surface formed by rotating a circle about any of its diameters. The center of the sphere is the center of the rotated circle. A sphere is the union of all points in space which are the same distance from a fixed point or center.



O is the center of the sphere.

$\overline{OA}$, $\overline{OB}$, and $\overline{OC}$ are radii of the sphere.

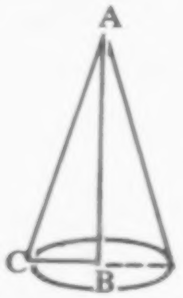
Cylinder: A right circular cylinder is a closed surface formed by rotating a rectangle around one of its sides. The bases are circular regions in parallel planes. The surface other than the bases is called the lateral surface. An *element* of the cylinder is a line segment perpendicular to the two bases with endpoints on the circles.



ABCD is a rectangle

E is an *element* of the cylinder.

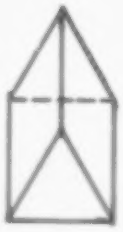
Cone: A right circular cone is a closed surface formed by rotating a right triangle about one of the rays forming the right angle. The rays forming the right angle are known as legs of the right triangle. The base of a right circular cone is a circular region.

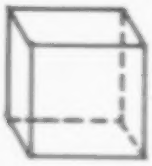


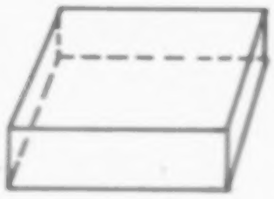
A is the *vertex.*

ABC is a right triangle, $\overline{AB}$ is one of its legs.

Prism: A prism is a closed surface with at least two congruent faces lying in parallel planes. All other faces are parallelograms.

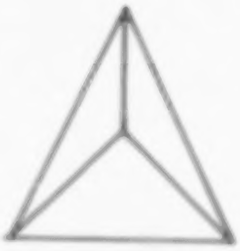




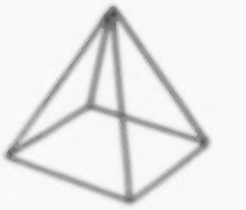


Pyramid: A pyramid is a closed surface whose base may be any polygonal region and whose sides are triangular regions having a common vertex.

Pyramids are named according to the polygon which forms the base.



triangular pyramid



square pyramid

Space Region—(Solid Region)

A space region is the union of the closed surface and its interior set of points.

TEACHING SUGGESTIONS FOR GEOMETRY

Basic Ideas of Geometry

The ideas of point, line, line segment, ray, plane, angle, and space should be developed using models from the physical world.

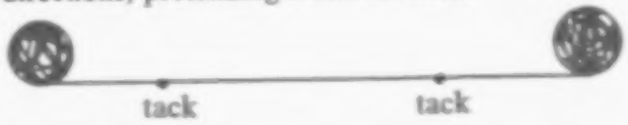
Point: The idea of a point may be introduced by discussing such questions as:

What brings the idea of a point to your mind? (Tip of a pencil, knife, scissors, thumb tacks, pins, nails, corner of a book)

How can the idea of a point be shown on paper?

Which dot is the best representation of an exact location in space?

Line: String can be used to illustrate the idea of a line by beginning at each end and rolling the string into two balls, attaching mid section to a board with thumb tacks as illustrated, and unrolling the string in both directions, pretending it has no ends.

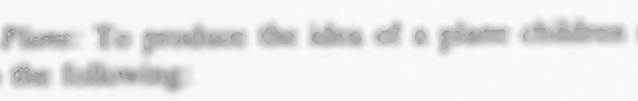


Line segment: The idea of a line segment is illustrated above by the tacks and the string between them.

The "lines" on the tablets of the students are pictures of the idea, line segment, with edges of paper as the end points.

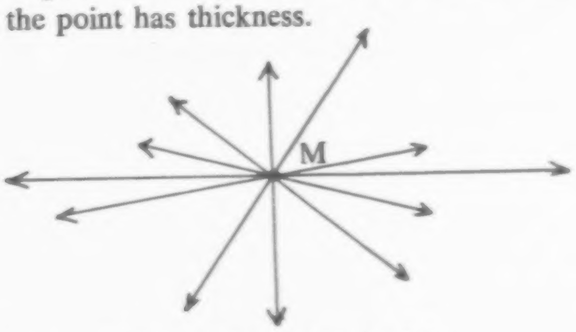
Ray: A seamstress' tape or a surveyor's tape may give the idea of a ray. The case is the end point with the tape extending from it in the one direction illustrating the ray.

The light sent out from the bulb (end point) of a flashlight can produce the idea of a ray. A ball of yarn and three children may also demonstrate the idea. The first child holds the end of the yarn, the second holds the yarn at a point to draw it taut, as the third child moves off unwinding the ball as he goes, pretending that it has no second end.



Plane: To produce the idea of a plane children may do the following:

Draw a picture of a point, label it M and discuss whether the point has thickness.



Draw pictures of lines through point M.

Discuss questions such as:

How many lines can pass through point M?

Could the paper be covered with pictures of lines through point M?

Can the set of points be said to cover the top of the desk if pencil point is placed at M and the paper turned around it?

Does a line have thickness?

Is a plane flat?

Does a plane have thickness?

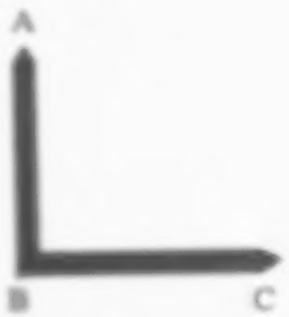
Place pencil point through the center hole of ruler (a number line), keep ruler flat on the desk, and spi it. Discuss whether this gives idea of a plane.

Hold pencil with point straight up, place ruler o pencil tip and spin ruler. Does this action define plane? (Yes)

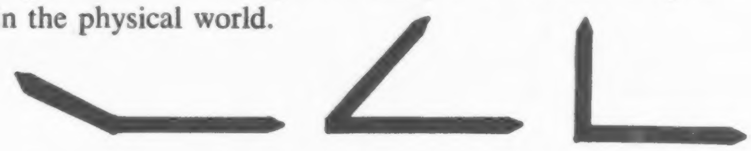
Tilt pencil and spin ruler. Does this action defin the same plane? (No)

Angle: Understanding of angles can be developd through use of a model prepared by children in the folowing manner:

Cut two strips of tagboard about $\frac{1}{2}$ inch wide and as long as desired. Join these strips at one end with a brass brad. Draw a ray down the center of each strip from the brad (B). Label a point somewhere along each strip. The brad represents the endpoint of the rays and vertex of the angle.



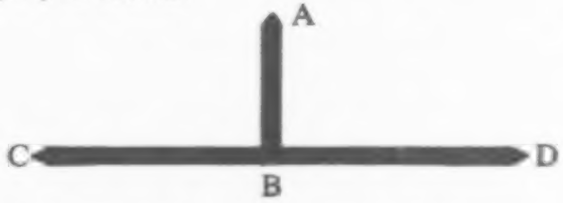
The model can be used to show the special kinds of angles most commonly used today and to locate angles in the physical world.



One ray may be placed along one side of the desk and the other fitted around the corner. Corners of rooms, doors, books may also be explored as children see other places that are models of angles. The position of the rays which seems to be the most often used can be observed and discussed. A model such as this is exemplified in a pair of scissors.

Right angle: A model for right angles can be prepared by:

1. Cutting two strips of tag board of different measure.
2. Fastening them together by placing a brad through the center of the longer one and through the end of the shorter one.
3. Labeling the vertex B (the brad) and points A, C, D, as shown.



Ray BA can be moved so that point A is closer to point C than to point D. When ray BA is upright so that the angles CBA and DBA are the same, they form right angles. Children may use this model to find where right angles are used in construction, art, printing of books, and other places.

A model of a right angle may also be produced by:

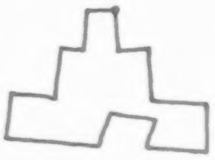
1. Folding and creasing a sheet of paper.
2. Folding the folded edge along itself.

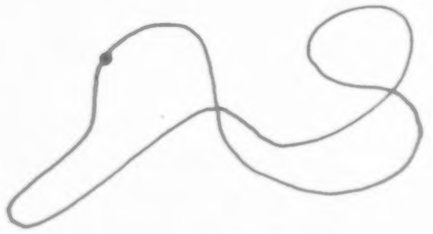
The corner at the fold will represent a right angle.

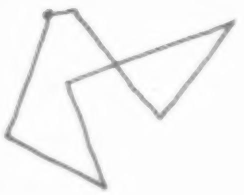
Path or curve: Paths through the woods represent sets of points traveled in moving from one place to another. Children may see paths or curves in the track of cars and trucks, creek beds, spilled paint, and snail tracks.

The idea of a path or curve may be demonstrated with finger paint as children dip a finger into the paint and make a series of dots on the paper . . . . or make one dot and draw the finger from the dot across the page leaving a path behind. The path represents a set of points. Paths may be wiggly or straight but all represent geometric curves.

Closed path or closed curve: Since a closed path or curve is a path which begins at one point and ends at the same point, children may illustrate the idea by drawing many different closed paths. Interesting designs may be made and examples collected from magazines.







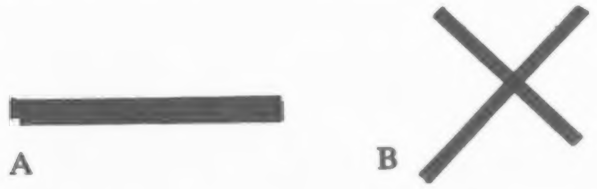
Simple closed curves: Simple closed curves are curves that do not cross themselves. Children may exhibit designs illustrating simple closed curves. Some letters of the alphabet and some numerals may be drawn as simple closed curves similar to the *m*.

Relationships

1. INTERSECTION

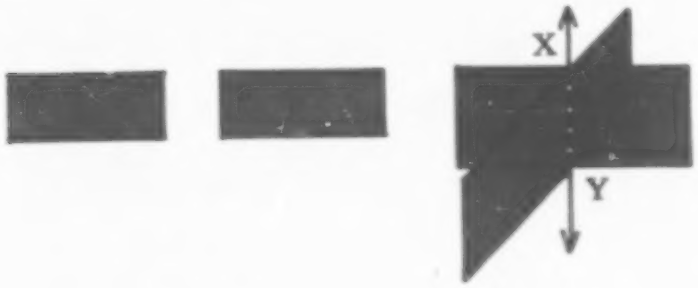
Ideas of intersection may be reviewed by recalling activities listed under the intersection of sets, page 18 of Sets and Numbers.

Intersecting lines: Strips of tagboard to represent lines may be used to illustrate the intersection of lines. Each child may experiment to find positions in which the strips can be made to intersect.



Position A shows that all points intersect. Points have no thickness and the two lines are the same line. Position B shows that the two lines have just one point in common.

Intersecting planes: To make a physical model of intersecting planes sheets of paper may be cut as shown and interlocked. It can be seen that the points at which the planes intersect become line XY. Children may discuss the possibility that a third or even more planes may intersect at line XY. (See illustration page 105.)



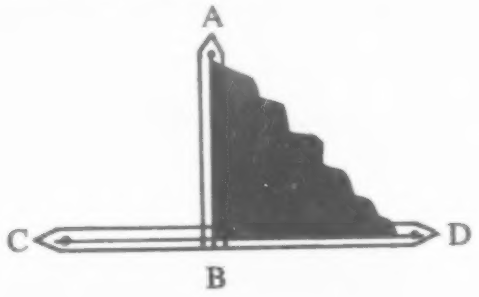
The idea that many planes may intersect at one line may also be illustrated by drawing a line on the chalkboard, labeling points on the line, and holding a sheet of cardboard representing a plane against the line in many positions.

If a third point not on the line is drawn, it becomes evident that the cardboard may contain all three points in only one position. One and only one plane may contain three points not in the same line.

Line intersecting a plane: Children may show the three relationships of a line and a plane by using a double pointed knitting needle to represent a line (a set of points) and a sheet of paper representing a plane (a set of points). When the line passes through the plane, the intersection is a point. When the line lies on the plane, the intersection is the line itself. When the line lies off the plane and there is no intersection, the line and the plane are parallel.

2. PERPENDICULARITY

Perpendicular lines: The perpendicularity of lines may be demonstrated by applying the folded paper model of a right angle to the model described for right angles on page 116. It can be seen from the illustration at the right that the corners formed by the intersection of two lines are right angles.



Line perpendicular to a plane: A line perpendicular to a plane may be demonstrated by:

a. Using a sheet of paper to represent a plane

b. Drawing intersecting lines through point Q on the plane

c. Holding a pencil at point Q

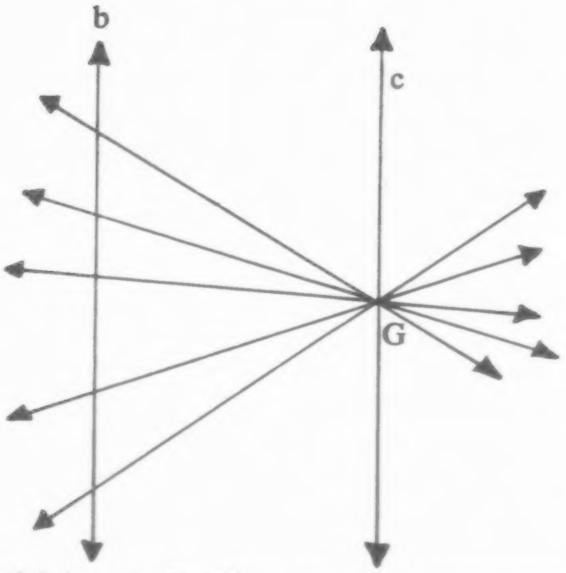
d. Placing folded paper model of right angle in angles formed by pencil and each line passing through point Q on the plane

e. Discussing relationship of pencil to the lines and to the plane.

Perpendicular planes: A model for perpendicular planes may be found in classrooms. The folded paper model for right angles may be applied to corners formed by walls. If the model fits exactly, the angles of the walls are right angles and the planes of the walls are perpendicular to each other. The model may be applied in many corners.

3. PARALLELISM

Parallel lines: Physical models of parallel lines may be observed in Venetian blinds, railroad tracks, tire tracks, edges of tape or line segments on notebook paper. Other examples may be identified by children.

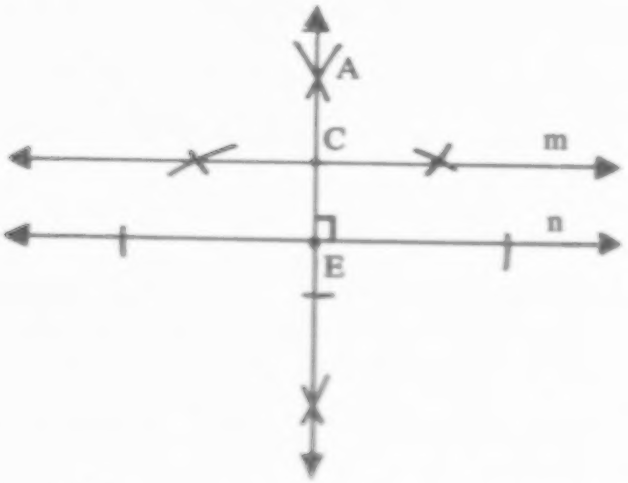


A model for parallel lines may be constructed by:

a. Drawing line *b* and point G off the line

b. Drawing many lines through G intersecting line *b*

c. Observing that lines through G almost cover the plane but leave a space, a missing line *c* which has no points on line *b*. Lines *b* and *c* are parallel lines and their intersection set is empty.



Parallel lines may also be constructed using a compass and straight edge as follows:

a. Draw line *n*.

b. Label a point E.

c. Construct a right angle at E. (See construction of right angles below.)

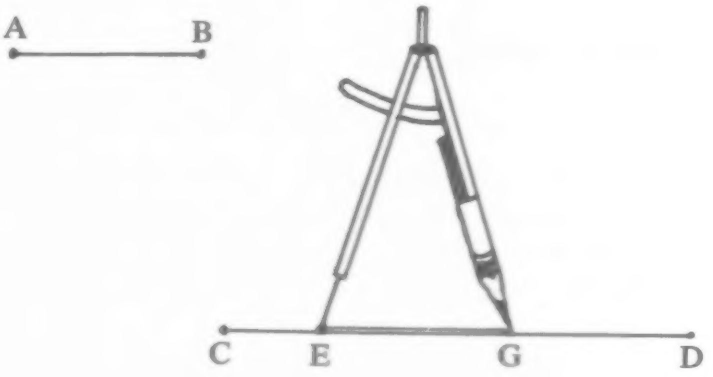
d. Construct a line perpendicular to ray EA at point C.

e. Label line *m*.
Line *m* and line *n* are parallel.

Parallel planes: Physical models of parallel planes may be seen in stacks of paper, opposite sides of rectangular boxes, shelves of book cases, or ceiling and floor. Additional models may be listed and discussed by children, applying the idea that parallel planes never intersect no matter how far extended.

4. CONGRUENCY

Models of congruent figures may be seen in sheets of paper from a notebook, floor tiles, stars in a flag, checkers, dominos, or playing cards. Others may be listed and discussed keeping in mind that congruent figures have the same size and shape.



Line segment: Line segments with end points equally far apart may be constructed using a compass by

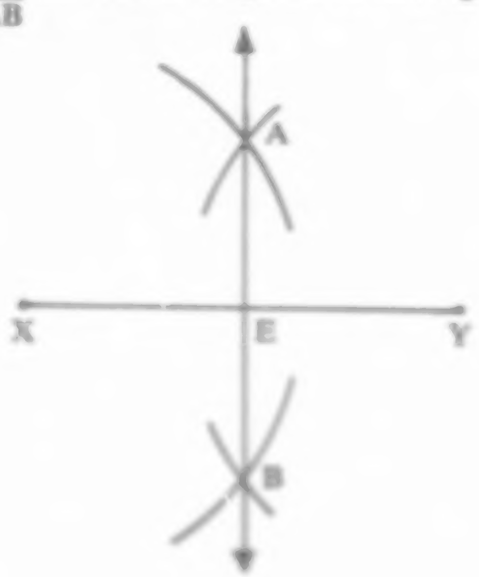
a. Drawing line segments AB and CD

b. Placing compass point on A and opening compass so that pencil point is on B

c. Placing point of compass on point E on $\overline{CD}$ with setting unchanged

d. Marking point G with pencil.

Line segment EG is congruent to line segment AB. $\overline{EG} \cong \overline{AB}$



Congruent line segments may also be constructed by *bisecting*, dividing into two congruent line segments, a given line segment. The following procedure may be used:

a. Draw line segment XY.

b. Open compass a distance greater than $\frac{1}{2}$ the measure of $\overline{XY}$.

c. With X and Y as centers, draw intersecting arcs above and below $\overline{XY}$.

d. Label points of intersection A and B.

e. Draw line through points A and B.

f. Label intersecting point of $\overline{AB}$ and $\overline{XY}$, E.

$\overline{XE} \cong \overline{EY}$

Since angles formed by the bisector of the line segment in the construction above are right angles, the same procedure may be used to construct:

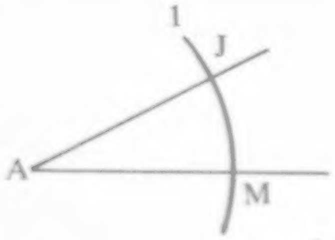
a. right angles

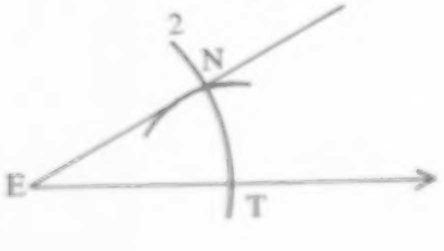
b. a line perpendicular to another line

c. a perpendicular bisector of a line.

Angles: Models of congruent angles may be seen in the many right angles used in construction of buildings or weaving of materials, in the corners of hexagonal tiles or points of a star.

Models of different shaped figures may be cut from construction paper and angles checked for congruency by applying one angle to another to match vertices and rays.





Understanding of congruency may be further developed by the construction of congruent angles using the following procedure:

a. Draw angle A.

b. Draw ray and label endpoint E.

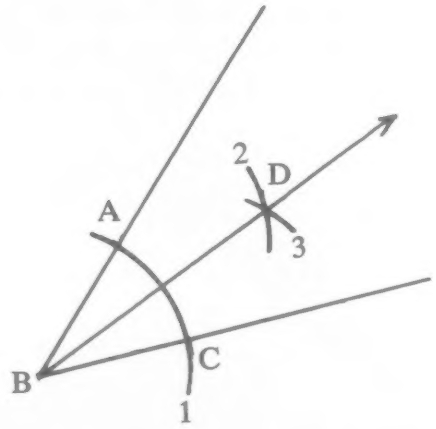
c. With A as center draw arc 1 intersecting rays of angle A at J and M.

d. Without changing opening of compass and with E as center, draw arc 2 intersecting the ray at T.

e. With T as center and opening of compass the distance between J and M, draw arc intersecting arc 2 at N.

f. Draw a ray from E through point N.

$\angle NET \cong \angle JAM$



Congruent angles are also formed when an angle is bisected. This may be done by using the following procedure:

a. Draw angle B.

b. With vertex, point B, as center, draw arc 1 intersecting the rays of the angle.

c. Label points of intersection A and C.

d. With A as center draw arc 2.

e. With compass opening unchanged and C as center, draw arc 3 intersecting arc 2.

f. Label point of intersection D.

g. Draw ray BD.

Ray BD bisects the angle ABC.

$\angle ABD \cong \angle DBC$

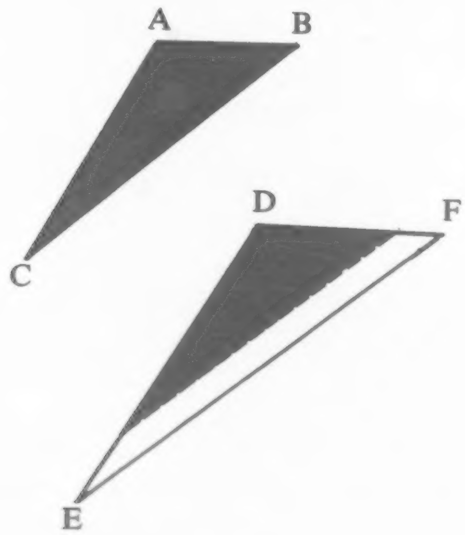
Plane figures: Models of congruent figures may be seen in stacks of paper, stacks of index cards, pages of a book or calendar.

Duplicate models of many different figures may be cut from cardboard and shuffled in a box or on bulletin board. Different activities in which congruent figures are matched may be designed.

5. SIMILARITY

Geometric figures that are similar have the same shape but not necessarily the same size. Children may cut from construction paper geometric shapes such as triangles, squares, circles, rectangles, quadrilaterals, and others. They may observe that while all squares have the same shape and all circles have the same shape, all triangles and all rectangles do not.

Children may experiment with ideas of similarity by cutting two triangles exactly alike. They may discover that, by cutting a strip parallel to any side of one of the triangles, the figure, although smaller, is of the same shape as the uncut triangle. The triangles, therefore, are similar but no longer the same size since the corresponding sides of one triangle have been shortened.



$\overline{AB}$ corresponds to $\overline{DF}$

$\overline{BC}$ corresponds to $\overline{FE}$

$\overline{AC}$ corresponds to $\overline{DE}$

It can likewise be discovered that these corresponding sides are in proportion to each other:

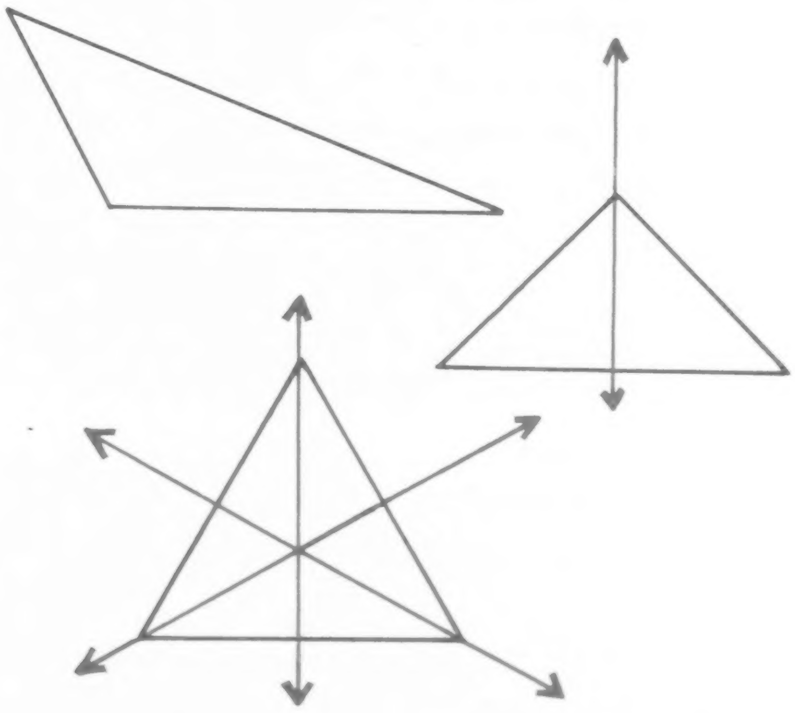
$\overline{AB}$ is to $\overline{DF}$ as $\overline{BC}$ is to $\overline{FE}$ as $\overline{AC}$ is to $\overline{DE}$.

Children may also discover that the angles of the smaller triangle will fit over the angles of the larger triangle. Other figures may be tested for similarity in the same way. For use of similar plane figures in indirect measurement, see page 138.

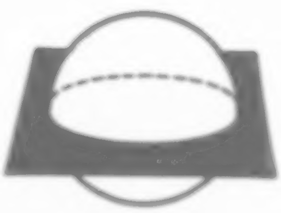
6. SYMMETRY

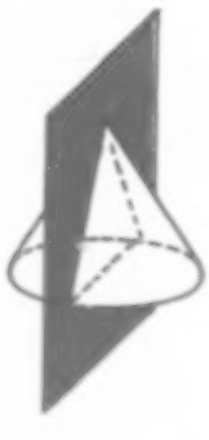
Line symmetry: A plane figure which is symmetrical may be divided by a line so that one half is an exact image of the other. Small hand mirrors with one straight edge are useful in the study of symmetry. Objects of various kinds can be tested by placing the mirror in such a way that the reflection is an exact image.

The line of symmetry marks points between the object and the reflected image. Different kinds of triangles and other plane figures may be examined for axes of symmetry as illustrated.



Plane symmetry: A symmetrical space figure may be divided by a plane so that one half is an exact image of the other. Models of many different space figures may be constructed or collected and examined for planes of symmetry. Through discussion it can be established that while some figures may have no planes of symmetry, others have one or more.

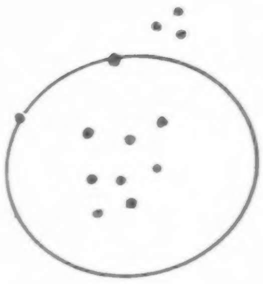




Plane Figures—Closed Paths

A rubber band, a piece of string with ends tied to form a ring, strings of beads may be used to illustrate the idea that a simple closed curve separates the plane into three sets of points—those on the interior, those on the curves, and those on the exterior.

A game of marbles with a ring drawn on the ground uses the ideas. The marbles on the interior are to be played. Those on the ring are neither on the interior nor the exterior set of points.



1. **CIRCLE**

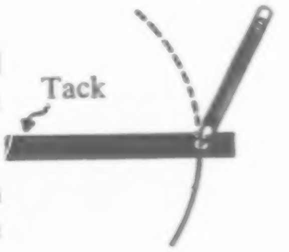
Places where the idea of a circle is used may be listed and discussed: cups, wheels of bus, end of pencil, telephone dial, holes in the dial, clock face, screw head, basketball hoop.

Children may experiment with ways to draw circles without tracing circular objects. A compass may be improvised by:

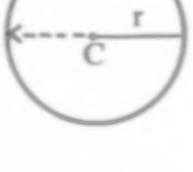
a. Cutting a hole in one end of a strip of tag board

b. Fastening the other end to a flat surface with a tack

c. Placing a pencil through the hole and rotating the strip around the tack.



The tack represents a point in the interior, the center, and the strip of tag board represents a line segment, the radius of the circle. By extending the radius through the center to a point on the curve, the new line segment becomes the diameter.



Other illustrations of the idea of center, radius and diameter, as well as of chord and arc, may be seen on page 110.

2. **ELLIPSE**

From discussions and pictures or diagrams of satellites in orbit, children have gained impressions of the idea of an ellipse.

A picture of an ellipse may be drawn by:

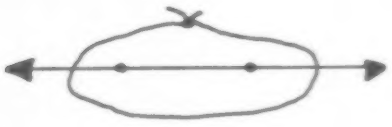
a. Placing two thumb tacks on a line

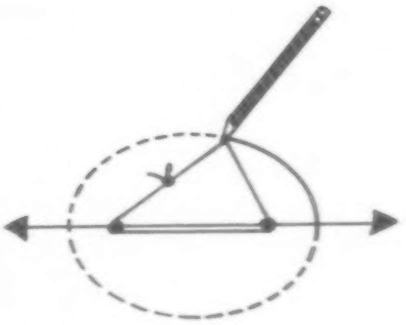
b. Forming a ring of string and placing it around the tacks

c. Inserting a pencil in the ring

d. Pulling the string taut and drawing a closed curve.

The resulting figure should be an ellipse with tacks being focus points. See also page 111.





3. **POLYGON**

Many examples of polygons may be seen in the physical world. Models of polygons may be constructed from strips of tag board, pipe cleaners, and drinking straws.

Strips of tag board may be fastened together with brads to experiment with the number of strips necessary to form a simple closed path. Strips of the same length and of different lengths may also be used in the experiment. Such questions as the following may be used to guide discussion of figures:

What kind of space figure is represented by one strip?

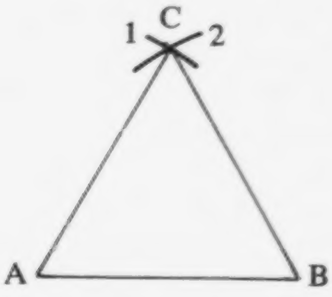
What plane figure is formed by two strips?

What is the least number of strips necessary to represent a closed figure?

What kind of figure is formed when all strips have the same measure?

What may be noted about angles when all sides are of the same measure?

Triangle: It has been noted that three line segments form a triangle. Experiments may be performed using brads and strips of tag board to construct triangles with all sides the same measure, with two sides the same measure, and with all three sides having different measure. Children may learn to identify triangles named in terms of the sides. See page 111.



Understanding of construction of congruent line segments may be used to construct equilateral triangles. The procedure follows:

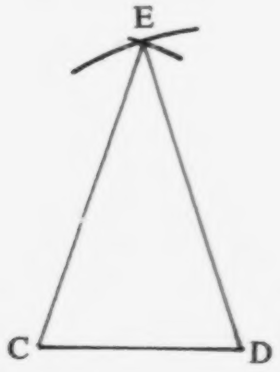
a. Draw line segment AB.

b. With compass opening equal to the measure of $\overline{AB}$ and with A as center, draw arc 1.

c. With B as center, draw arc intersecting arc 1 at C.

d. Draw line segments AC and BC.

$\triangle ABC$ is an equilateral triangle since $\overline{AB} \cong \overline{AC} \cong \overline{CB}$.



Isosceles triangles may also be constructed using ideas related to congruent line segments. The procedure follows:

a. Draw line segment CD.

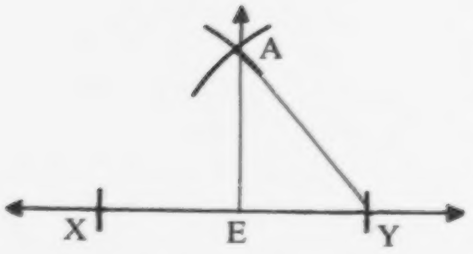
b. Open compass a distance greater than $\frac{1}{2}$ the measure of $\overline{CD}$.

c. With C and D as centers, draw intersecting arcs.

d. Label point of intersection E.

e. Draw line segments CE and DE.

$\triangle CDE$ is an isosceles triangle since $\overline{CE} \cong \overline{DE}$.



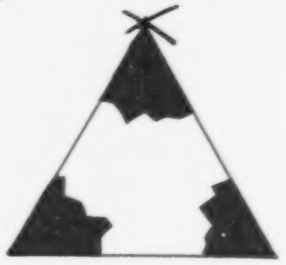
A right triangle may be produced by using the procedure for constructing a right angle and continuing as follows:

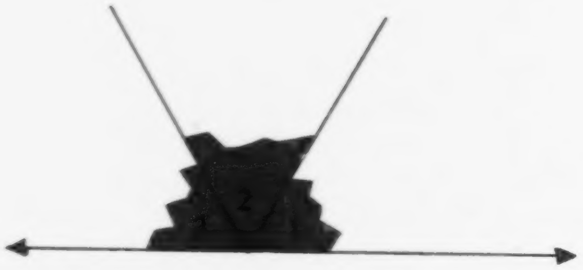
Draw line segment AY

$\triangle AEY$ is a right triangle since $\angle YEA$ is a right angle.

Experiments may be extended to include identification by angles by:

a. Cutting models of equilateral triangle, tearing off the vertices to fit them along a line as illustrated





b. Repeating the experiment using isosceles and scalene triangles

c. Discussing the relationship of angles in any triangle to a line.

Quadrilateral: Four strips of tag board fastened with brads may be used to develop ideas of quadrilaterals by:

Using all strips of different lengths

Using two strips of the same length and two different lengths

Fastening the strips described above in different combinations

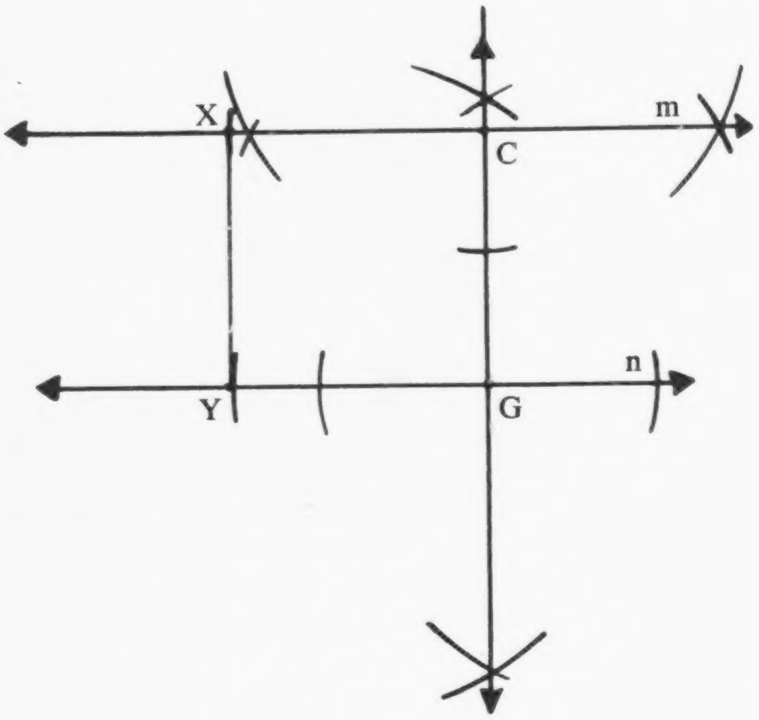
Using two strips of one measure and two longer strips to experiment with combinations

Using all strips of the same measure

Pushing opposite corners to skew figures into different shapes

Testing for right angles using folded paper model.

Children may learn to identify figures formed according to sides and angles.



Understanding of the relationship of parallel lines may be used to construct a square with straight edge and compass as follows:

a. Draw two parallel lines by constructing right angles.

b. Open compass the distance of the measure of $\overline{CG}$.

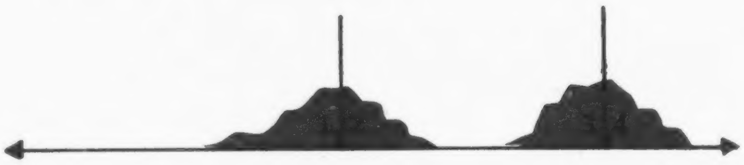
c. Mark off arcs along line *m* from point C and along line *n* from point G.

d. Label points of intersections X and Y.

e. Draw line segment XY.

Closed curve GCXY is a square since all sides are congruent, opposite sides are parallel and angles are right angles.

Relationship of angles of quadrilaterals to a line segment may be determined in the same manner as for a triangle.



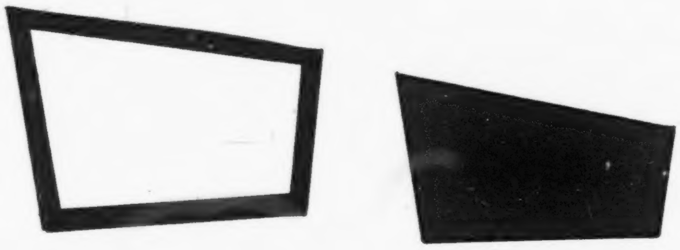
Pentagon: Five strips of tag board may be used to develop ideas of pentagons in the same manner as described above.

Hexagon: Six strips of tag board may be used to develop ideas of hexagons in the same manner as described above.

Octagon: Eight strips of tag board may be used to develop ideas of octagons in the same manner as described above.

Plane Region

When the idea that a simple closed curve separates a plane into three sets of points is well established, children should have little difficulty understanding a plane region as the union of the set of points on the curve and the set of points on the interior. To illustrate the idea models of plane figures cut from paper may be contrasted with models constructed from strips of tag board.



Space Figures—Closed Surfaces

Physical models of closed surfaces can be seen in inflated balloons, footballs, and basketballs; empty boxes including rectangular prisms, cylinders, pyramids, cubes, and other shapes. It can be seen that the balloon separates the space into three sets of points; the interior, exterior, and the balloon itself, or the surface.

Children may collect boxes of interesting shapes to be used in learning to describe space figures in precise geometric terms.

Sphere: Questions such as the following can be used to develop ideas of spheres:

Is a football a closed surface? Is it a sphere? Why?

Which of the following might be considered examples of closed surfaces: baseball, tennis ball, ping pong ball, golf ball? Why?

Which of the balls listed above most nearly represent spheres? Why?

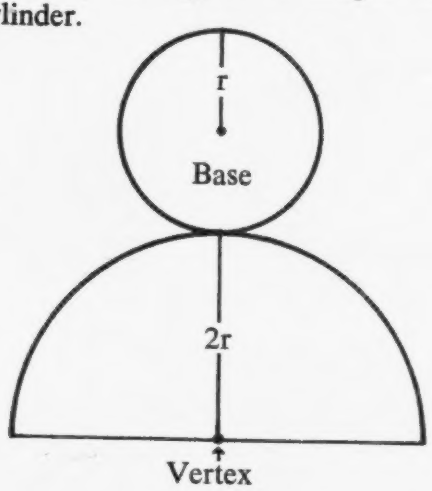
Is a globe representing the earth a sphere? Why?

Children may cut out the model of a circle and spin it about a diameter to see the relationship between a circle and a sphere. Since the circle will spin on any of its diameters, it can be seen that the diameters of the sphere and diameters of the circle are the same and that radii of the sphere and radii of the circle are the same.

Cylinder: Physical models of cylinders can be found easily: unopened cans, oatmeal boxes, mailing tubes, and cylinders from motors. A cardboard model of a rectangle may be rotated around one of its sides to demonstrate the relationship of cylinder to rectangle. It can be seen that the height of the cylinder is the same as the length of the side of the rectangle, that the base is a circular region with a radius of the same measure as the other side of the rectangle. This idea may be illustrated also by wrapping flexible wire around a cylinder, then wrapping wire over the ends of the cylinder and discussing:

The shape of the resulting figures

Relationship of figures to height and diameter of cylinder.



Pattern for Constructing a Cone

Cone: One model, the ice cream cone, is familiar to most children. Models may also be constructed using the accompanying illustration. Discussions based on activities similar to the following may also serve to extend ideas related to cones:

Spinning a coin, describing the figure formed and discussing relation of coin and sphere

Spinning a rectangle around one side, describing the figure formed and discussing relation of rectangle to cylinder

Spinning a right triangle around one of the rays forming the right angle, describing the figure formed and discussing relation of triangle to cone.

Prism: Collections of boxes of many sizes and shapes can be used in developing ideas of prisms. As each box is examined, such questions as the following may be used:

Is the figure a closed surface?

How many faces has it?

How many pairs of faces lie in parallel planes?

How many faces are congruent?

Is the figure a prism?

Pyramid: Pictures of ancient pyramids may be discussed to introduce the idea that the faces of all pyramids are triangular regions that meet at a common point or vertex. As children experiment with constructions of models it can be seen that any regular polygonal region may be used as the base of a pyramid. Such questions as the following may guide discussion of figures constructed:

What is the shape of the base?

What is the shape of each face?

May any face of the pyramid be named as base? Why?

May any corner of the pyramid be named as vertex? Why?

What is the name of the pyramid constructed?

Space Region—(Solid Region)

Use of the term "solid region" may be helpful in introducing the idea of a region that is the union of the closed surface and the interior set of points. Discussions in relation to closed surfaces may be reviewed to establish an understanding of the difference between ping pong and golf balls, tennis and baseballs. Glass paper weights in many shapes in contrast with glass bottles or jars may be used to extend understanding of the idea. A styrene foam ball and old tennis ball may be sliced in half and nature of the interior examined and discussed to insure that children have clear understanding of the differences between closed surfaces and space regions.

MEASUREMENT

Measurement, or the process of measuring, associates a number with a physical object or a geometric figure to show quantity or amount. Measurement developed historically through needs of Man to compare and describe quantitative aspects of his environment. The continuing need for measurement has led to the development of more precise measuring instruments and to applications in diverse situations.

One of the first ideas about measurement that a child experiences is the comparison of physical objects that tells him which is longer or shorter, larger or smaller, taller or shorter. Later the child learns to choose a unit and associate numbers with the objects.

Important concepts related to all measurement are:

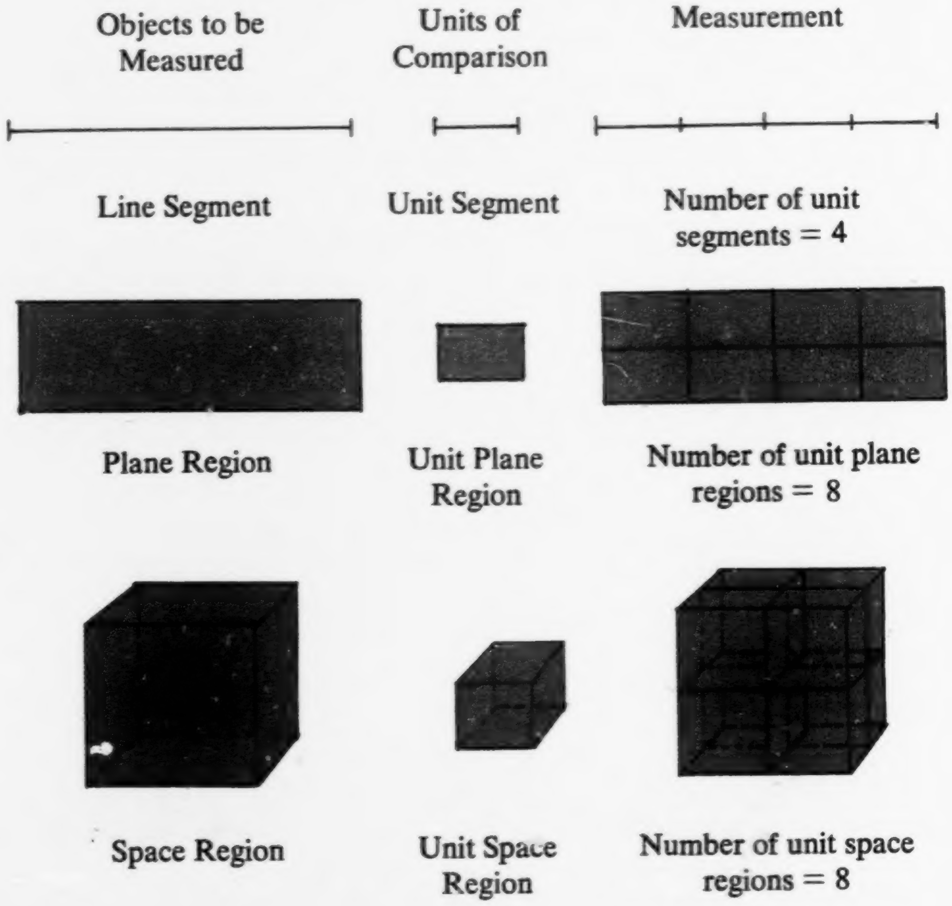
The meaning of measurement
The arbitrary nature of units of measure
The necessity for standard units of measure
The relation of the property of the object to be measured and the unit of measure used
The approximate nature of measurement of physical objects.

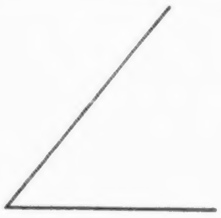
Measurement is the most widely used area of applied mathematics in the elementary school curriculum. Ideas of measurement are involved in the sequential development of sets of numbers, the number line, and geometry. Measurement provides teachers with many opportunities for practical application of mathematical ideas, for practice in operations on numbers, and for active participation and experimentation.

Mathematical Ideas	Illustrations and Explanations

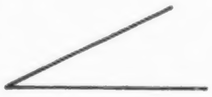
Basic Ideas of Measurement

A *measure* is the number obtained by comparing an object with a *unit of measure* which is of the same nature as the object.

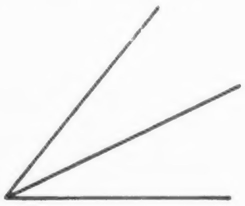

Objects to be Measured
Units of Comparison
Measurement
Line Segment
Unit Segment
Number of unit segments = 4
Plane Region
Unit Plane Region
Number of unit plane regions = 8
Space Region
Unit Space Region
Number of unit space regions = 8



Angle



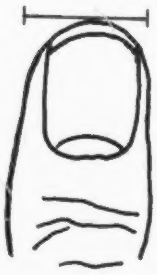
Unit Angle



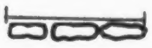
Number of unit angles = 2

Standard Units of Measure: Standard units of measure are selected arbitrarily by agreement. The history of measurement is the story of the need for and the development of standard units of measure. The history of measurement provides one vehicle for developing understanding of the necessity for standard units of measure.

Width of thumb



Three grains of barleycorn



Marks on a metal bar

Each type of measurement may be thought of as having a standard unit arbitrarily selected. For example: meter or yard for length, gram or pound for weight, second for time, and degree for angle.

Other Units

$\frac{1}{3}$ yard = 1 foot, 1760 yards = 1 mile

$\frac{1}{16}$ pound (lb.) = 1 ounce (oz.), 2000 lbs. = 1 ton

$\frac{1}{60}$ degree = 1 minute, 360 degrees = 1 revolution

$\frac{1}{1000}$ second = 1 millisecond, 60 seconds = 1 minute

Other units may be defined in terms of the standard, as either a multiple or as a fractional part of the basic unit.

Equivalents

$5\frac{1}{2}$ ft. = ☐ in.	12 in. = 1 ft.
$5\frac{1}{2} \times 12 = 66$	$5\frac{1}{2}$ ft. = 66 in.
80 oz. = ☐ lbs.	16 oz. = 1 lb.
$\frac{80}{16} = 5$	80 oz. = 5 lbs.

Direct and Indirect Measure: Direct measurements are made by comparing a unit of measurement with the object.

A measuring tape is used to determine length of a classroom.

A measuring cup is used to determine the amount of flour for a cake recipe.

Indirect measurement is a measure obtained by calculation in situations where direct comparison of a standard unit of measure and the object is not possible or practical.

To measure rise or fall of temperature, the expansion or contraction of a liquid in a closed glass tube is measured. (Liquid thermometer) The rise or fall of temperature is indicated by an appropriately calibrated scale attached to the tube.

To measure the passage of time, a clock or watch, which is a mechanical device of gear wheels driven by electricity or springs, is used.

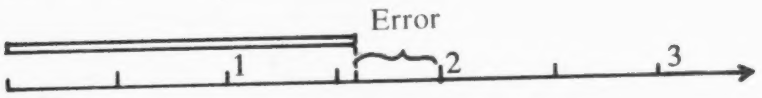
To measure the distance a child lives from school, the number of revolutions of the wheel of an automobile may be counted. A series of appropriate gears enables the distance to be indicated on the automobile odometer.

To measure the area of a rectangular region, the lengths of adjacent sides are measured and the area is determined by an appropriate mathematical formula.

Approximate nature of measurement:

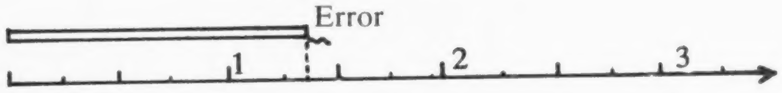
All measurements determined by the use of a measuring instrument are approximate because of the characteristics of measuring instruments themselves and of human limitations in the use of the tools.

Since measurements are made to the nearest given unit, an *error* is always involved. This error is no more than one half the unit of measure being used.



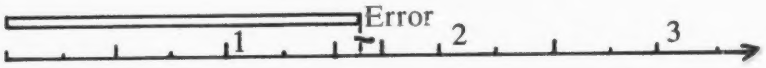
Length recorded as 2 inches to the nearest inch.

Error is no more than one-half inch.



Length recorded as $1\frac{1}{2}$ inches to the nearest half-inch.

Error is no more than one-quarter inch.



Length recorded as $1\frac{3}{4}$ inches to the nearest quarter inch.

Error is no more than one-eighth inch.

Precision: The precision of a measurement is determined by the unit of measure used.

Given Measurement	Precision of Measurement
$8\frac{3}{8}$ in.	$\frac{1}{8}$ in.
3.6 miles	0.1 miles
2 lbs. 4 oz.	1 oz.
3 tons	1 ton

A measurement is only as precise as the measuring instrument used: The smaller the unit of measure, the more precise the measurement. The degree of precision desired and the size of the object to be measured determine the appropriate unit of measure.

Object	Precision of Instrument	Appropriate Instrument
Book	$\frac{1}{2}$ in.	Foot ruler
Rug	1 ft.	Yard stick
Distance between cities	1 mi.	Odometer

Types of Measurement: Different types of measurements are needed for describing different types of objects, quantities, or conditions. Each type may include more than one unit of measure.

SYSTEMS

	English	Metric
Liquid	teaspoon, tablespoon, cup, pint, quart, gallon	milliliter, liter, kiloliter
Dry	teaspoon, tablespoon, cup, pint,* quart,* peck, bushel	milliliter, liter, kiloliter
Weight	ounce, pound, ton	milligram, gram, kilogram
Time	second, minute, hour, day, week, month, year, decade, century	same
Temperature	degree, Fahrenheit	degree, Celsius
Counting	unit, pair, dozen, gross, ream	same
Angular	second, minute, degree, revolution	same
Length	inch, foot, yard, rod, mile, light year	millimeter, centimeter, meter, kilometer
Area	square inch, square foot, square yard, acre	square millimeter, square centimeter, square meter, hectare
Volume	cubic inch, cubic foot, cubic yard	cubic millimeter, cubic centimeter, cubic meter

Many industries, trades, and occupations have units of measure particularly their own.

*Not equivalent to the same unit of liquid measurement.

	Pint	Quart
Liquid measure	$28\frac{7}{8}$ cubic inches	$57\frac{3}{4}$ cubic inches
Dry measure	$33\frac{3}{5}$ cubic inches	$67\frac{1}{5}$ cubic inches

The Metric System

The Metric System of measure is a decimal system since relationships between units are based on powers of ten.

Relationships of Metric Units

	Linear	Weight Mass	Volume
1000 units	kilometer	kilogram	kiloliter
100 units	hectometer	hectogram	hectoliter
10 units	decameter	decagram	decaliter
Standard: Unit	Meter	Gram	Liter
$\frac{1}{10}$ unit	decimeter	decigram	deciliter
$\frac{1}{100}$ unit	centimeter	centigram	centiliter
$\frac{1}{1000}$ unit	millimeter	milligram	milliliter

The English System

In the English System units of measure are selected arbitrarily with no consistent pattern of relationships between units.

Relationships of Units of English System

1760 yds. = 1 mi.

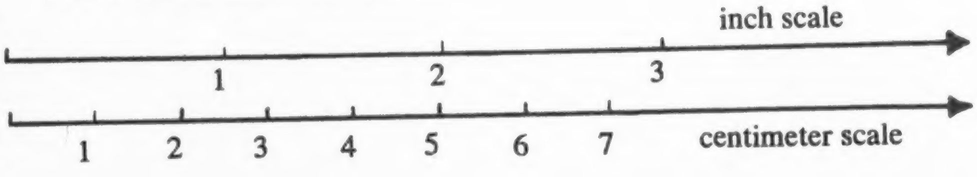
$5\frac{1}{2}$ yds. = 1 rd. 2000 lbs. = 1 T. 4 qts. = 1 gal.

Standard:

Yard	Pound	Quart
$\frac{1}{3}$ yd. = 1 ft.	$\frac{1}{16}$ lb. = 1 oz .	$\frac{1}{2}$ qt. = 1 pt.
$\frac{1}{36}$ yd. = 1 in.		$\frac{1}{4}$ qt. = 1 cup

Relationship of metric units to English units: Since it is necessary at times to convert measurements from English to metric, or metric to English systems, it is desirable to develop understanding of an approximate relationship between the more commonly used units.

One inch is about 2.5 centimeters



Excessive computation is to be avoided since the principle idea of relative relationships is often lost in this manner.

One kilometer is a little more than one-half mile.

One kilogram is a little more than two pounds.

One meter is a little more than one yard.

The metric system is used in many countries and by scientists in all countries. Many industries in the United States now use the system since it is easier to convert from one unit to another in a base ten system. Legal units of measure in the United States have been determined in terms of the meter, gram, and liter since 1866.

Converting from one unit to another: A necessary phase of the study of measurement is learning how to convert a given measurement to an equivalent measurement expressed in a different unit of measure. The objective is to learn a systematic pattern based on understanding which will enable the student to convert units successfully.

Two approaches for converting units are described below:

The relationship between 2 measures can be expressed as a ratio. This fact can be used in converting from 1 unit to another. For example:

foot to inches is 1:12 or $\frac{1}{12}$

inches to foot is 12:1 or $\frac{12}{1}$

ounces to pounds is 16:1 or $\frac{16}{1}$

inches to centimeters is 1:2.54 or $\frac{1.00}{2.54}$

In a proportion the numbers must always be from the same denomination. This fact can be used in converting from one unit to another. If a given measurement of 60 inches is to be converted to feet, a proportion may be used:

$$\frac{1}{12} = \frac{\square}{60} \text{ or } 1:12 = \square:60$$

Solving the equation results in $\square = 5$ and 60 inches equals 5 feet.

If a given measurement of $5\frac{1}{2}$ pounds is to be converted to ounces, a proportion may again be used:

$$\frac{1}{16} = \frac{5.5}{\square}$$

Solving the equation results in $\square = 88$ and $5\frac{1}{2}$ pounds equals 88 ounces.

A second approach to converting units depends to some extent on experience and common sense.

To change from a large unit to a smaller unit, multiply by the number that tells how many smaller units are equivalent to one large unit.

Change 24 yards to feet

3 feet = 1 yard $24 \times 3 = 72$ 24 yards = 72 feet

Change 47 kilometers to meters

1000 meters = 1 kilometer $47 \times 1000 = 47{,}000$

47 kilometers = 47,000 meters

To change from a small unit to a larger unit, divide by the number that tells how many smaller units are equivalent to one large unit.
Change 488 ounces to pounds

1 pound = 16 ounces $488 \div 16 = 30\frac{1}{2}$ 488 ounces = $30\frac{1}{2}$ pounds

Change 36 kilometers to miles

1 mile = 1.61 kilometers $36 \div 1.61 = 22.4$

36 kilometers = 22.4 miles

Estimation of Measurement

Estimation is the mental process for arriving at a reasonable measurement by seeing an object in proper perspective to a unit of measure. Estimates are based on known facts or past experience. Experience and mental computation are essential in arriving at a satisfactory result when actual measurement is not feasible or not needed.

Using parts of students' bodies as units of measure: finger joints, hand span, foot.

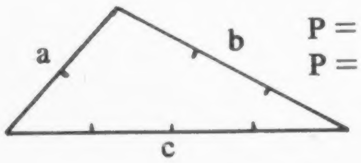
Estimating measures

Stepping or pacing to estimate distance.

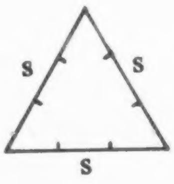
Estimating number of discrete objects

Perimeter

The *perimeter* of a simple closed path is the distance traveled in moving from one point along the path back to the starting point.

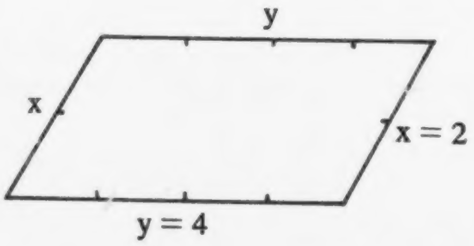


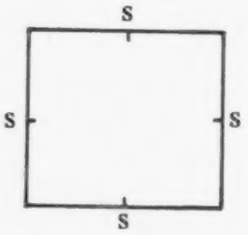
$P = 2 + 3 + 4$
$P = a + b + c$



$P = 3 + 3 + 3$
$P = s + s + s$

Polygon: The perimeter of a polygon is the sum of the measures of the line segments which form its sides.





$P = 2 + 4 + 2 + 4$
$P = (2 \times 2) + (2 \times 4)$
$P = 2y + 2x$ or $2\,(l + w)$

$s = 2$
$P = 2 + 2 + 2 + 2$
$P = 4s$

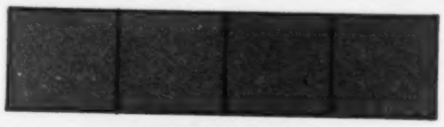
Circle: The perimeter of a circle, conventionally called the *circumference,* is the product of *pi* and the measure of a diameter of the circle.

The constant relationship which exists between a diameter of a circle and its circumference may be demonstrated in the following way:

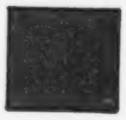
1. Wrap string around model of a circle.
2. Stretch string out straight.
3. Cut strip of paper the same length as a diameter.
4. Compare length of diameter with length of string. Result of comparison: the circumference is a little more than 3 times the diameter.
5. More approximations of the relationship of the diameter to the circumference will give $3\frac{1}{7}$ or 3.14.
6. This constant ratio, with value approximately $3\frac{1}{7}$ or 3.14 is known as π, the Greek letter pi.
7. $C = \pi d$
   $C = 2\pi r$

Area

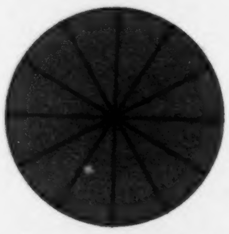
The *area* of a given plane region is the number of unit plane regions required to cover the plane region with no overlapping. Plane region—See page 123, Geometry.



Area is 4 Units

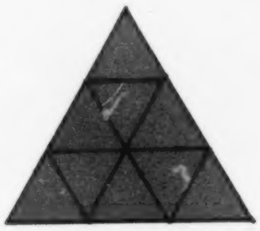


Unit

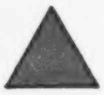


Area is 12 Units

Unit



Area is 9 Units

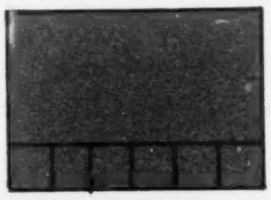


Unit

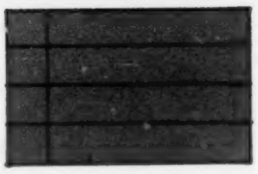
Although any plane region can be used as the unit, a square plane region such as a square inch or a square centimeter is usually used as the standard unit of area measurement.

1. AREA OF POLYGONS

Rectangular region: The area of a rectangular region is the product of the measures of its length and width.



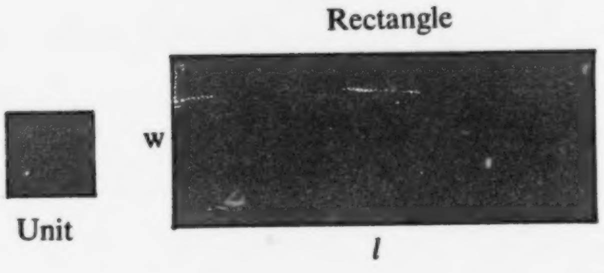
Count number of square units which can be placed along one side of rectangle.



Count the number of rows.

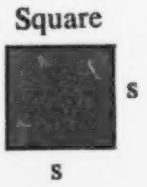
The area is the product of the number of rows and the number of units in each row.

$A = 4 + 6 = 24$ square units



There are l square units along one side, and w rows of the units.

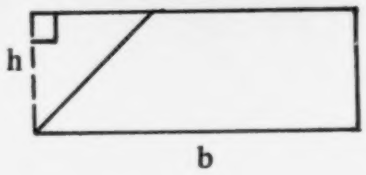
$A = l \times w$



$A = s \times s$ or s^2

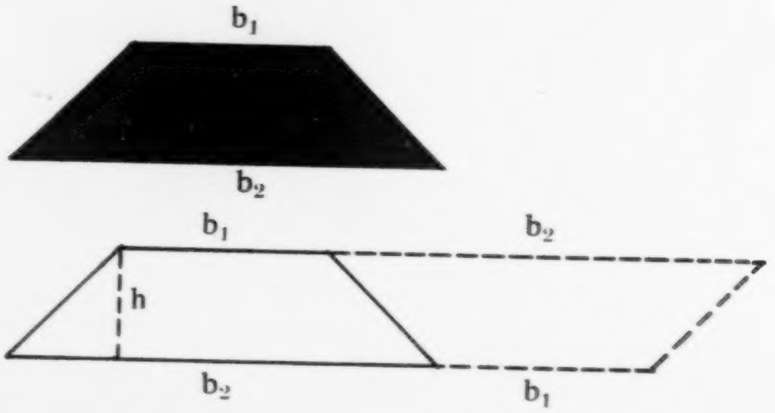
Parallelogram region: The area of a parallelogram region is the product of the measures of its *base* (length) and height (*altitude* or width).

Cutting along a line perpendicular to the bases, a parallelogram region can be made into a rectangular region.



$A = b \times h$

Trapezoidal region: The area of a trapezoidal region is one half of the product of the measures of the height (altitude) and the sum of the measures of its parallel sides (bases).

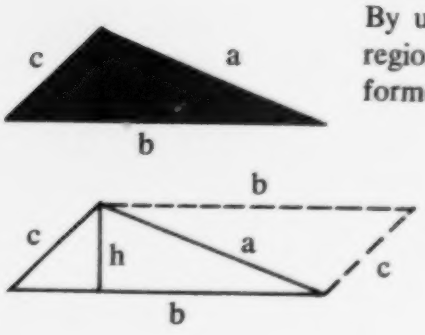


By adding a congruent trapezoidal region, the above parallelogram region can be made with base $(b_1 + b_2)$.

Area of the above parallelogram region $= h \times (b_1 + b_2)$.

Thus, area of trapezoidal region $= \frac{1}{2} \times h \times (b_1 + b_2)$.

Triangular region: The area of a triangular region is equal to one half the product of the measure of its base and height.



By using another congruent triangular region, a parallelogram region may be formed.

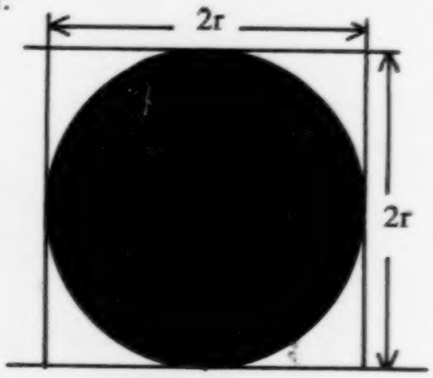
The area of the parallelogram region $= b \times h$

Hence, the area of the triangular region may be expressed:

$$A = \frac{1}{2} b \times h$$

Circular region: The area of a circular region is equal to the product of pi and the square of its radius.

An approximation for the area of a circular region may be developed by using the following model:



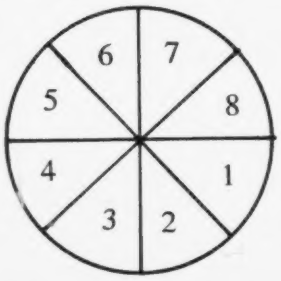
The area of the square region is:

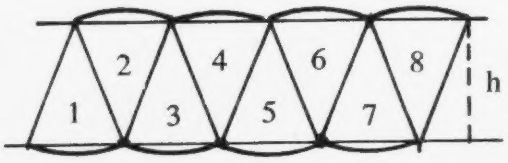
$2r \times 2r = (2r)^2 = 4r^2$

Hence, the area of the circle is approximately $3r^2$

$A \approx 3r^2$ ($\approx$ means "approximately equal to")

The following model may also be used to develop the formula for finding the area of a circular region.





Cut a circular region into an even number of congruent wedge shaped sections. Fit these together to form an approximation of a parallelogram region; the smaller the wedges the better approximation.

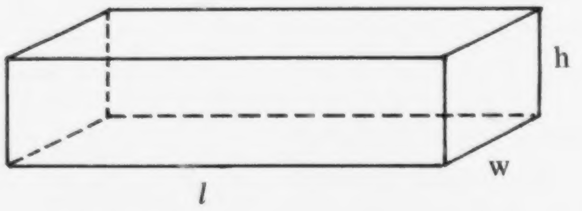
If the region were a parallelogram, its base would be equal to $\frac{1}{2}$ of the circumference of the circle and its height equal to the radius of the circle:

$C = 2\pi r$ $b = \frac{1}{2}C$ or πr $h = r$

$$\begin{aligned} A &= b \times h \\ &= \pi r \times r \\ &= \pi r^2 \end{aligned}$$

2. AREA OF CLOSED SURFACES:

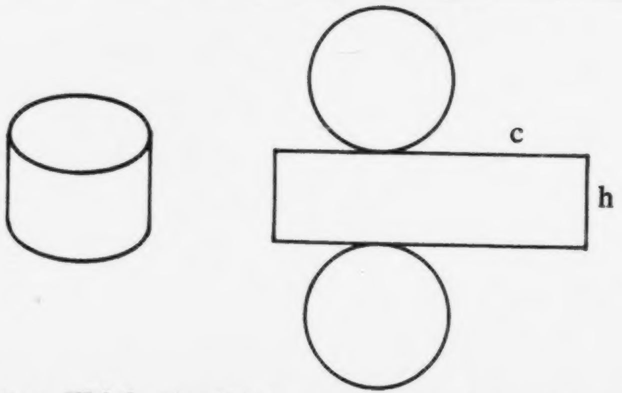
Prism and pyramid: The surface area of a prism or pyramid is found by adding the measures of the areas of the *faces* or plane regions which make up the closed surfaces.



Total Area $= 2 \times (l \times w) + 2 \times (w \times h) + 2 \times (l \times h)$

Total Area $= (2 \times B) + (h \times p)$ where B is the area of base, h is the height of the prism and p is the perimeter of the base.

Cylinder: The surface area of a cylinder is found by adding the area of its lateral surface and the areas of the two bases.



Lateral Area: Think of a label of a can cut in a line perpendicular to the bases. Stretched out the label makes a rectangular region with the same area as the lateral surface of the cylinder.

Lateral Area = c × h

$c = 2\pi r$ or πd

Lateral Area = $2\pi r \times h$ or $\pi d \times h$

Area of 2 bases = $2\pi r^2$

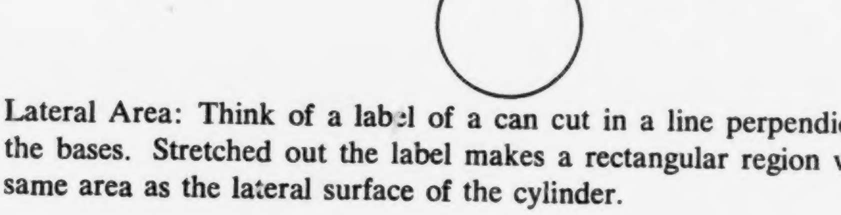
Surface area of a cylinder = lateral area + area of bases

$A = 2\pi r \times h + 2\pi r^2$ or

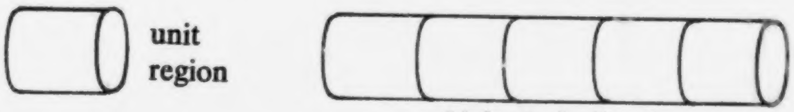
$A = 2B + (h \times c)$ when B is the area of one base, h is the height and c is the circumference of a base.

Volume

The *volume* of a space region is the number of unit space regions required to fill the space region.

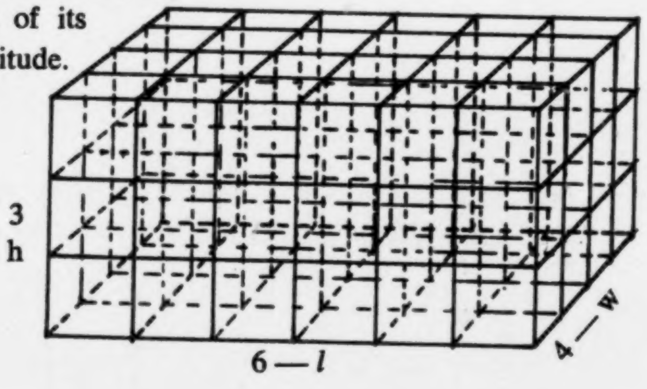


Volume is 4 units.



Volume is 5 units.

Rectangular prism: The volume of a rectangular prism is the product of the area of its base and the altitude or the product of its length, width, and altitude.



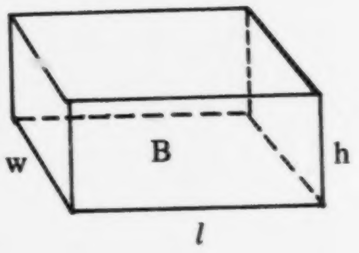
Find the number of cubic units of the base layer.

6 × 4 = 24

Hence, 24 cubic units will form the base layer.

The height tells the number of layers.

Hence, volume = 3 × 24 = 72 cubic units.

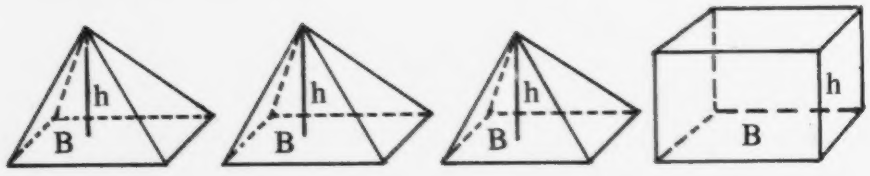


If B is the area of the base,

$$V = B \times h$$

$$V = (l \times w) \times h$$

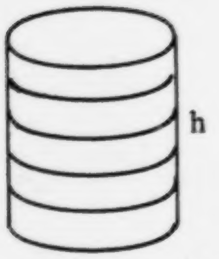
Regular pyramid: The volume of a pyramid is one third the product of the area of its base and the altitude or height.



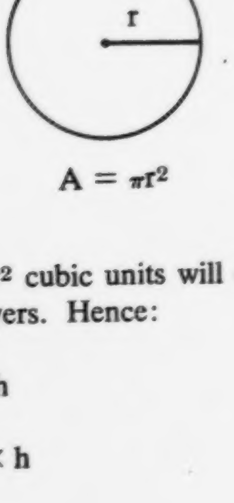
Given a prism and a pyramid with congruent bases and altitudes, the pyramid filled three times with sand will fill the prism.

$$V = \frac{1}{3} B \times h$$

Right circular cylinder: The volume of a right circular cylinder is the product of the area of its base and the altitude.



unit
region



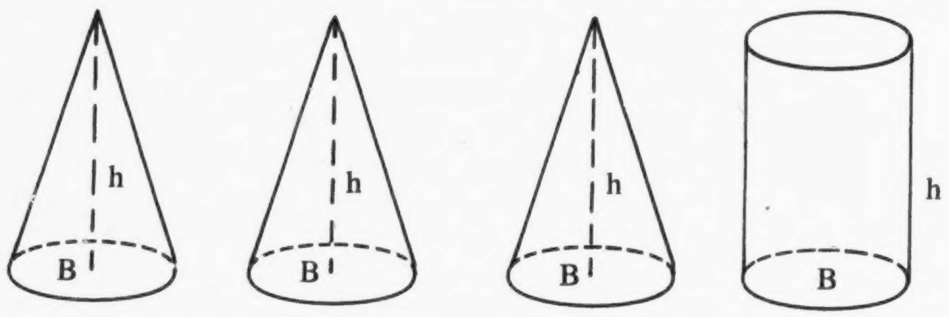
$$A = \pi r^2$$

The area of the base is πr^2. Hence, πr^2 cubic units will cover the base completely 1 unit deep. There are h layers. Hence:

$$V = B \times h$$

$$V = \pi r^2 \times h$$

Right circular cone: The volume of a right circular cone is one third the product of the area of its base and the altitude.



Given a cone and a cylinder with congruent bases and altitudes, the cone filled three times with sand will fill the cylinder. Hence,

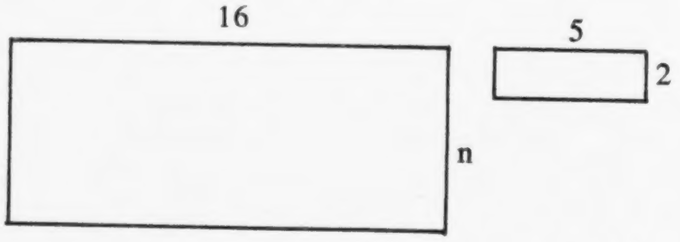
$$V = \frac{1}{3} B\,h \text{ or } V = \frac{1}{3}\pi r^2 h$$

B is the area of the base or πr^2.

> Measures of some geometric figures, such as the surface area or volume of a sphere, have not been included. Measures of such figures may be more appropriately introduced at the junior high or secondary levels.

Use of Similar Plane Figures in Indirect Measurement

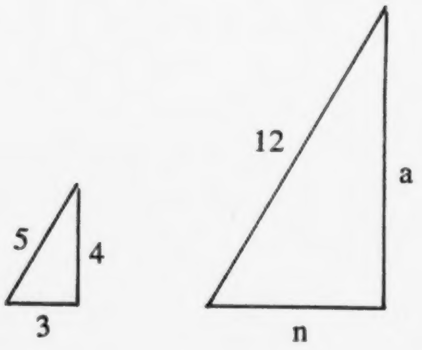
Use of similar polygons in indirect measurement depends upon understanding that corresponding sides of similar polygons are in proportion.



$$\frac{n}{2} = \frac{16}{5}$$

$$5 \times n = 16 \times 2$$

$$n = \frac{32}{5} = 6\frac{2}{5}$$



$$\frac{12}{5} = \frac{a}{4}$$

$$5 \times a = 4 \times 12$$

$$a = \frac{48}{5} = 9\frac{3}{5}$$

$$\frac{12}{5} = \frac{n}{3}$$

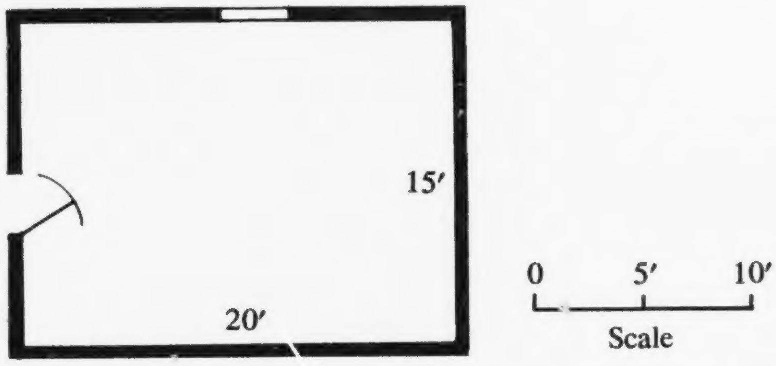
$$5 \times n = 12 \times 3$$

$$n = \frac{36}{5} = 7\frac{1}{5}$$

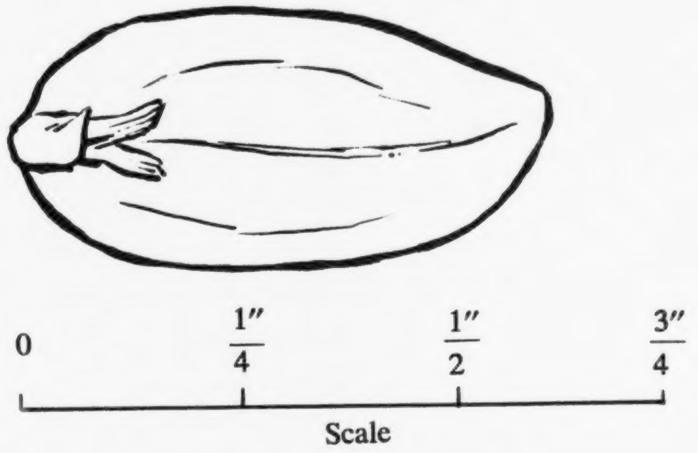
Scale Drawing

Scale drawing is a representation of an object equal in size, enlarged, or reduced to convenient or appropriate size. The drawing is similar to the original, therefore, all measurements are in proportion to those of the original object. The scale is the ratio of measurements on the object to measurements on the drawing.

A rectangular room which measures 15 feet by 20 feet may be represented by a drawing, $1\frac{1}{2}$ inches by 2 inches, where $\frac{1}{2}$ inch represents 5 feet.

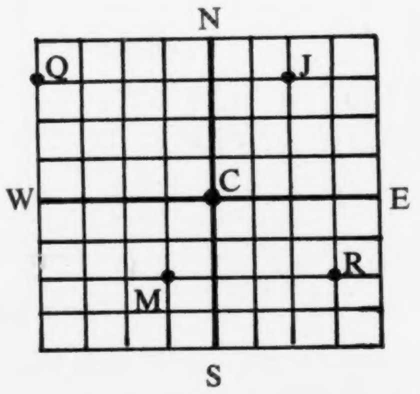


A peanut which measures $\frac{5}{8}$ inch may be represented by a drawing $2\frac{1}{2}$ inches where 1 inch represents $\frac{1}{4}$ inch.



Rectangular Coordinate System

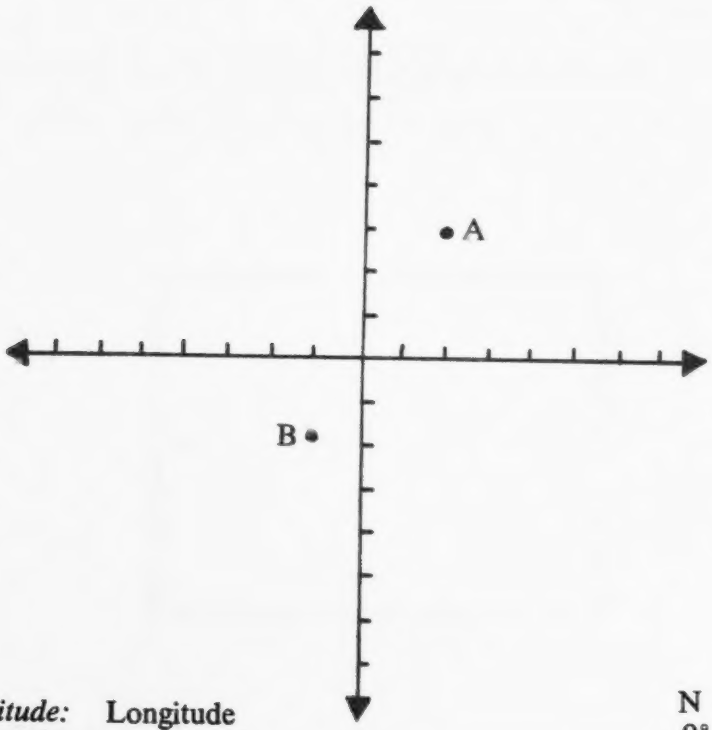
The *rectangular coordinate system* is made by the perpendicular intersection of two number lines. In the rectangular coordinate system, ordered pairs of numbers are used to locate points in the plane. There is a one-to-one correspondence between the set of points in the plane and the set of ordered pairs of real numbers.



Johnny lives 2 blocks east and 3 blocks north of the center of town. C is the center of town. His home is shown by J. Mary lives 1 block west and 2 blocks south of the center of town. Her house is shown by M.

Q: 4 blocks west, 3 blocks north

R: 3 blocks east, 2 blocks south



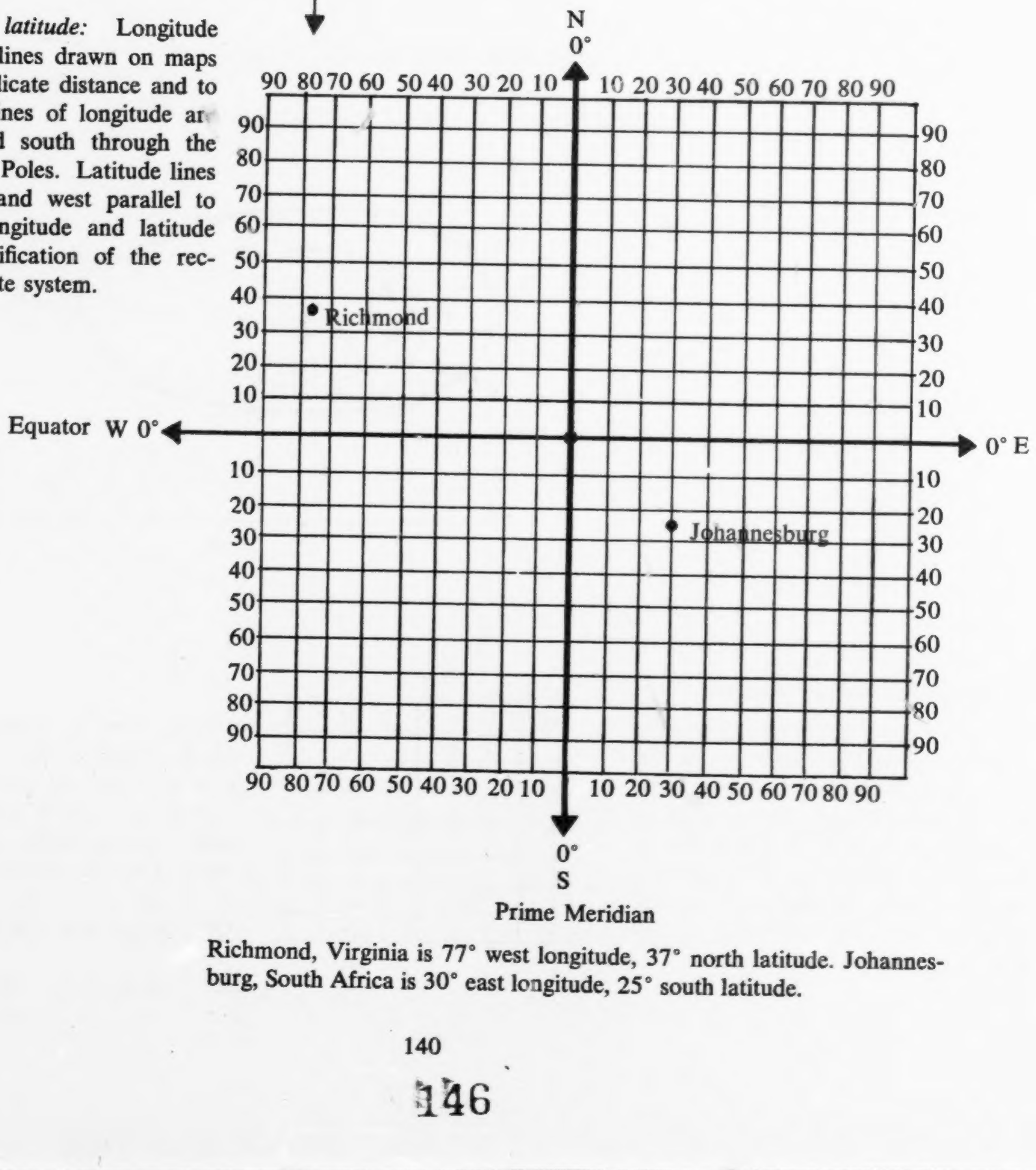
Locate the point that matches (+2, +3)

The point A (+2, +3) is located by moving to the right 2 and up 3.

Locate the point (−1, −2)

Point B, (−1, −2) is located by moving to the left 1 and down 2.

Longitude and latitude: Longitude and latitude are lines drawn on maps and globes to indicate distance and to locate points. Lines of longitude are drawn north and south through the North and South Poles. Latitude lines are drawn east and west parallel to the equator. Longitude and latitude represent a modification of the rectangular coordinate system.

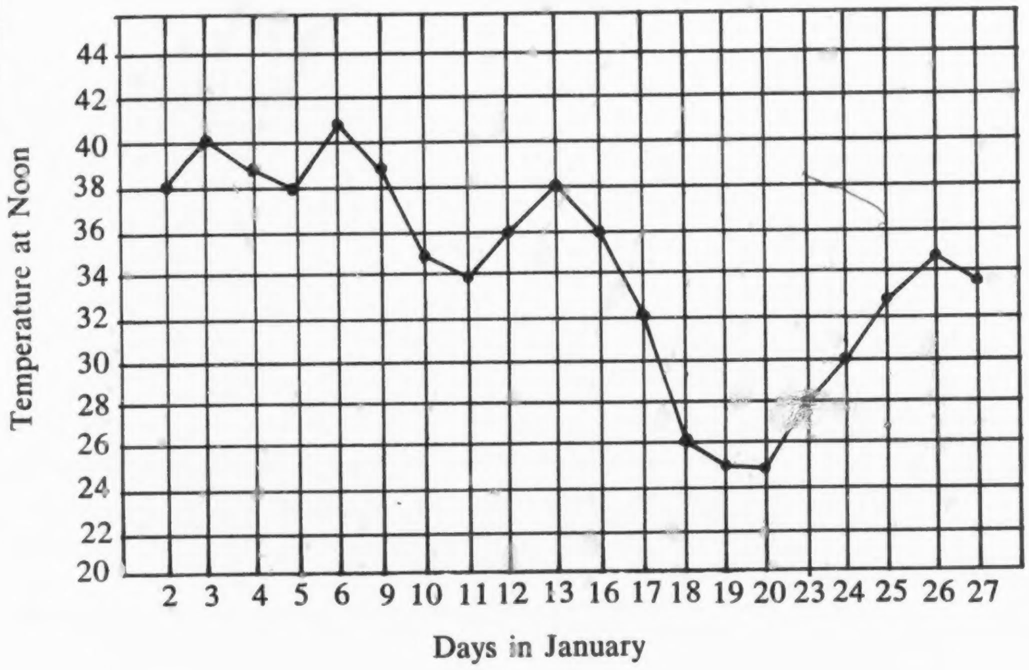

N
0°
90 80 70 60 50 40 30 20 10
10 20 30 40 50 60 70 80 90
Richmond
Johannesburg
Equator W 0°
0° E
0°
S
Prime Meridian

Richmond, Virginia is 77° west longitude, 37° north latitude. Johannesburg, South Africa is 30° east longitude, 25° south latitude.

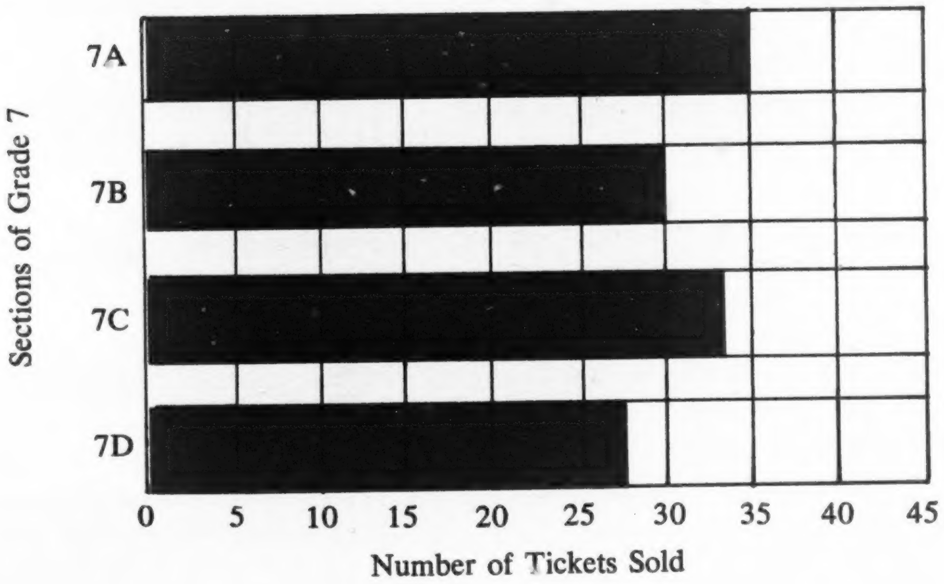
Graphs: Most graphs are based on the rectangular coordinate system or some adaptation of it. Many graphs will utilize only the first *quadrant* of the rectangular coordinate system since only positive numbers will be involved.

LINE GRAPH

Temperature at Noon Each School Day in January



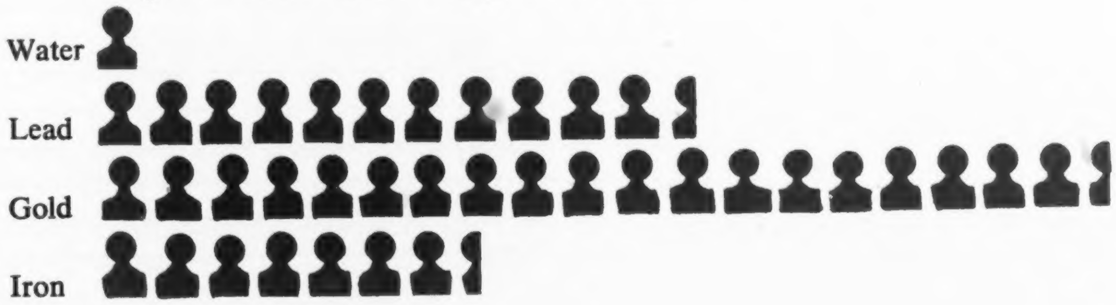
BAR GRAPH



Other types of graphs may also be used in representing data.

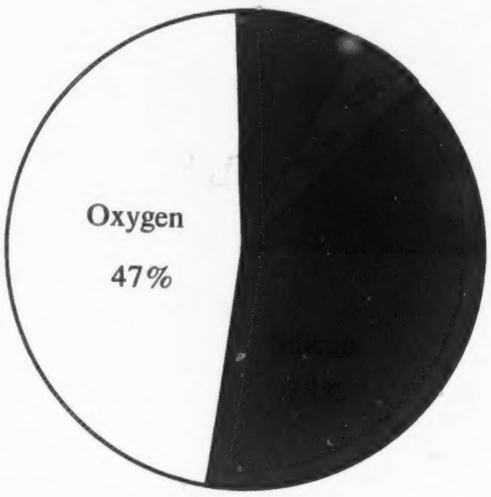
PICTOGRAPH

Weights of Some Metals Compared to Water



CIRCLE GRAPH

Elements in the Earth's Crust



> Graphs are particularly useful in statistics since relationships among items of data may be clearly and quickly seen and interpretation made simpler.

Introduction to Elementary Statistics

To answer many questions it is necessary to collect, analyze, and interpret facts. These facts are known as *data.* The study of numerical data is known as *statistics.*

Collecting data: The first step in a statistical study is the collection of information or data.

Organizing data: For better understanding, data are arranged in some systematic fashion. One way to do this is to write the data in order of size from the highest to the lowest, and then show how often each item has occurred. Such an arrangement is called a *frequency distribution.*

As part of the study of weather in a science class, students observe and record the outside temperature in degrees Fahrenheit each day at noon. During January the following temperatures were recorded:

	M	T	W	T	F
First Week	38	40	39	38	41
Second Week	39	35	34	36	38
Third Week	36	32	26	25	25
Fourth Week	28	30	33	35	34

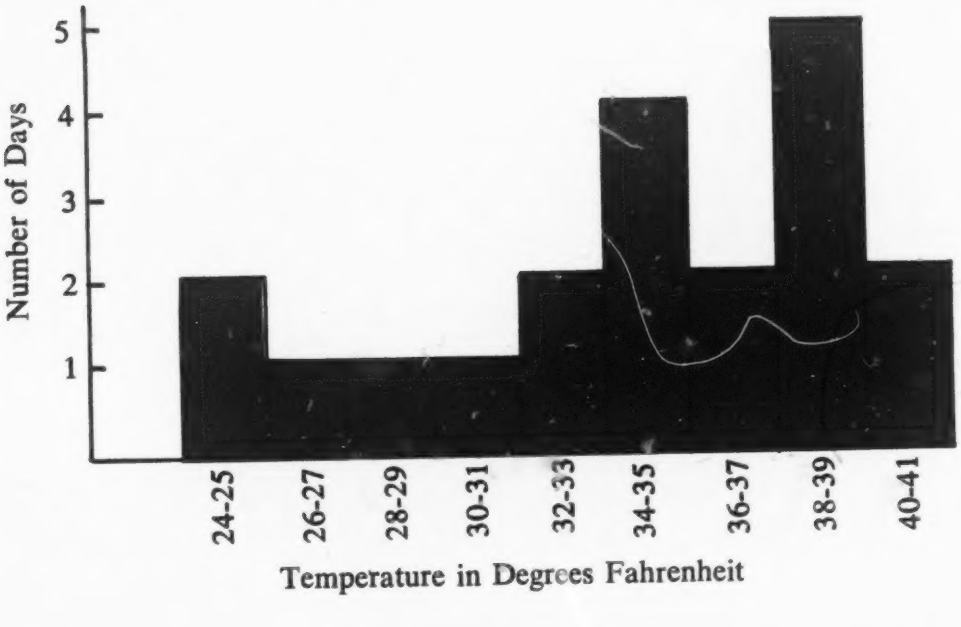
Temperature in Degrees Fahrenheit	Frequency
41	/
40	/
39	//
38	///
37	
36	//
35	//
34	//
33	/
32	/
31	
30	/
29	
28	/
27	
26	/
25	//
	Total 20

The data may consist of large numbers of items, or have items that differ so greatly in value that a frequency distribution showing each item becomes unwieldy. In such cases, the data are entered by intervals. This is called a *grouped frequency distribution*.

Temperature in Degrees Fahrenheit

Intervals	Frequency
40-41	2
38-39	5
36-37	2
34-35	4
32-33	2
30-31	1
28-29	1
26-27	1
24-25	2

The information in a grouped frequency distribution, as above, may be pictured in a vertical bar graph, called a *histogram*.



Instead of handling an entire table of data, it is often more convenient to use a number that is representative or typical of all the data in the table. Such numbers are called *measures of central tendency* since they tend to be descriptive of all the data. There are three such measures and each has advantages.

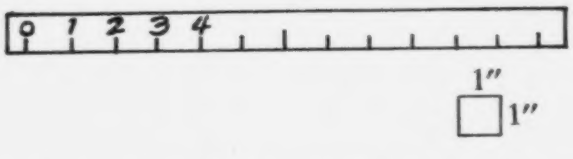
Mathematical Ideas	Illustrations and Explanations
The arithmetic *mean* (or *average*) of a set of data is equal to the sum of all the data values divided by the number of values.	The arithmetic mean for temperatures during five days of the first week of January (See table page 142) is the sum of those temperatures divided by the number of readings. $\frac{38 + 40 + 39 + 38 + 41}{5} = \frac{196}{5} = 39.2$ 39.2 is the mean or average temperature for five days in the first week of January. The arithmetic mean for the time during 20 days of January is the sum of temperatures for all 20 days (682) divided by number of readings (20). $\frac{682}{20} = 34.1$ 34.1°F is the mean or average temperature during twenty days of January.
The *median* of a set of data is the middle number of the set when the members are arranged in order of size. To find the median, first arrange the numbers in order of size. The median is the middle number if the set contains an odd number of numbers. If the set contains an even number of numbers, the median is a number halfway between the two numbers that are nearest the middle.	There are 20 numbers in the frequency distribution on page 142. Counting down from the top, or up from the bottom, the middle temperature is 35°F. Thus the median temperature for January is 35°F. Given the numbers 21, 23, 24, 28, 34, the median is 24. Given the numbers 62, 65, 70, 74, 78, 81, the median is 72.
The *mode* of a set of data is the number that occurs most frequently. To find the mode (of a set of data) arrange the data in a frequency table and select the item having the greatest frequency.	The temperature 38°F occurs most frequently in the frequency distribution on page 142. Thus 38°F is the mode for temperature during January.

TEACHING SUGGESTIONS FOR MEASUREMENT

Basic Ideas of Measurement

Children should have experience measuring with arbitrary units of measure such as, pencils, pieces of chalk, or edges of books to measure length or distance; sheets of paper, cards, tiles, or flat sides of books or blocks to measure surface; blocks, books, or boxes to measure capacity or volume. Attempts may also be made to measure a flat surface using circular or triangular regions, or to measure a circular region using square or rectangular regions or smaller circular units. If records are kept as children measure objects using arbitrary units, discussion of results can bring out the need for employing a unit which has the same nature as the object being measured. It may also point out the need for standard units which everyone uses and understands.

Standard Units of Measure: Each child may be provided with a strip of cardboard between 12 and 13 inches long and a smaller strip measuring one inch square. Starting at a point which is labeled 0 and using the 1 inch unit to mark unit segments, each inch is labeled. These rulers should be checked against other rulers and yard sticks to show that an inch is the same length no matter how long the measuring instrument may be. The self-made rulers may then be used to measure things at home and at school.



Standard units of measure of many kinds should be available in the classroom and opportunity to use them should be provided. Foot and metric rulers, yard and meter sticks, tape measures, teaspoons, tablespoons, balance scales and weights, units for liquid and dry measure, clocks, and large scale thermometers should be accessible to children at all times. Children should have experience filling half-pint, pint, quart, half-gallon, and gallon containers with sand or water as well as applying units of linear measure. Objects of different size but of the same weight should be available for comparison both with and without weighing. By applying standard units of measure to concrete materials, children discover relationships between cup, pint and quart, ounces and pound, meter and centimeter. If relationships discovered are recorded on charts and made accessible when needed, children learn equivalents in a meaningful way.

In addition to the classroom collection, each child should have available for individual use an appropriately marked ruler and, as the need develops, a protractor. In developing understanding of the need for standard units the cubit referred to in Biblical literature may be discussed. The length of a cubit was determined by the length of a man's arm from finger tip to elbow. The variation in cubit as represented by forearm measurements obtainable in the classroom can help children to understand why this unit is no longer in use.

Older children may study history of the development of instruments of measurement. This may range from research on primitive and archaic units, to explanation of modern devices.

Direct and Indirect Measurement: First experiences with measurement generally involve direct application of a unit to the object to be measured. Use of a ruler to measure length or a cup to measure liquid is direct measurement. Although use of the term is not stressed, the idea should be established.

Discussion of other ways to measure can result in a classroom collection of such instruments as the following: thermometer, speedometer, clocks, scales (platform, balance, and spring), barometer, gauges (air pressure, rain, wind, and steam), and meters.

Opportunities to use these instruments to gain greater understanding of indirect measurement should be provided. Large scale thermometers may be used to measure temperatures of various substances. A weather station may be visited to see gauges which measure and record temperature, barometric pressure, rainfall, and wind velocity. Devices for measuring rise and fall of the tide may be examined.

As formulae are developed and used in measuring area and volume of geometric figures, ideas of indirect measurement are extended.

Approximate Nature of Measurement: Such activities as the following may help children to understand the approximate nature of measurement:

> Recording and comparing results when several children measure the same object using different instruments of measure; such as, 6 inch ruler, 12

inch ruler, yard stick, or steel tape; tablespoon, cup, or pint measure.

Recording and comparing results when several children measure the same object using the same measuring instrument.

Comparing the amount of liquid in a glass measuring cup when the scale is read at eye level and below eye level.

Discussing such possible reasons for variations as: angle from which measurement is read, thickness of pencil mark, thickness of mark on instrument, dust or dampness on scales, human manipulation of the instrument.

Precision: Length and width of sheets of paper, desk tops, or any flat surface may be measured using rulers marked in 1 inch, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, and $\frac{1}{16}$ inch units. If results are recorded in a chart similar to the one below, it can be seen that precision of a measurement is determined by the unit of measure used and that as the size of the unit decreases the measurement approaches closer and closer the actual measure.

Size of Unit	Length of Object
1 inch	10 inches
$\frac{1}{2}$ inch	$9\frac{1}{2}$ inches
$\frac{1}{4}$ inch	$9\frac{3}{4}$ inches
$\frac{1}{8}$ inch	$9\frac{5}{8}$ inches
$\frac{1}{16}$ inch	$9\frac{11}{16}$ inches

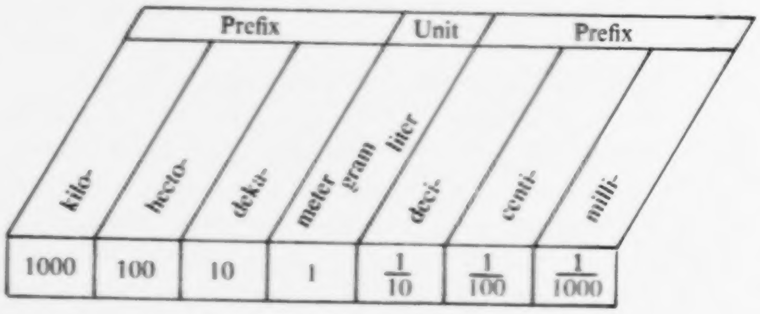
Types of Measurement: The importance of a collection of measuring devices readily available in each classroom cannot be over emphasized. It is through much contact with such devices that children acquire intuitive knowledge of the difference between pound and ounce, pint and quart for liquid and dry measure, quart and liter, square inch and cubic inch, and of the type of instrument appropriate for measuring each given substance.

Knowledge of less familiar types of measurement may be gained by keeping a record of types encountered in conversations, stories, newspaper articles, radio and television ads, or labels on packages. Types peculiar to special trades, industries, or occupations may be especially interesting. For example:

Unit of Measure	Trade or Occupation
fathom	maritime
knot	maritime
ream	paper
quire	paper
hogshead	tobacco, molasses
dram	drug
cord	wood
hand	horses
carat	gold, jewels
chain	surveyors

The Metric System

It is important to develop understanding that the metric system of measurement is the most used in the world today and is based on powers of ten. Its basic units are meter, gram, and liter. Blank charts similar to those used in developing understanding of place value and decimals may be made in the form of transparencies or wall charts. Units may be written in, and appropriate prefixes and numerical values added, as relationships are discussed.



Prefix			Unit	Prefix		
kilo-	hecto-	deka-	meter gram liter	deci-	centi-	milli-
1000	100	10	1	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$

As a part of the study of the history of measurement children may be interested in learning of the development of the metric system and the derivation of prefixes used.

The English System

Although the English system is the one most familiar to children in the United States, it may be necessary to emphasize the point that its units of measure were chosen arbitrarily over periods of time and are not related to a pattern as in the metric system. Charts similar to the one given on page 129 may be developed as units in the English system are used.

Relationship of Metric Units to English Units: Since excessive computation in converting from English to metric system or from metric to English may tend to obscure understanding of the approximate physical relationship, physical models should be used in the development of the ideas.

A yard stick may be measured against a meter stick to discover that a meter is a little more than a yard.

Liquid may be measured by using quart and liter measures to demonstrate that a liter is a little more than a quart.

Two parallel line segments of the same length may be marked, one in inches, the other in centimeters to show that one inch is equal to about $2\frac{1}{2}$ or 2.5 centimeters.

Some packaged food products, particularly those prepared abroad, are marked in metric units. Comparison may be made with similar products marked in the English system.

Converting From One Unit to Another: Two approaches for converting from one unit to another are described on page 130. Since understanding of either approach depends to a great degree on experience and common sense, many opportunities for the manipulation of concrete materials should be provided. Four one-foot rulers may be placed end to end and ways of determining the number of inches represented in the four feet discussed:

Inches may be counted, 48

Repeated addition may be used,

$12 + 12 + 12 + 12 = 48$

Multiplication may be used, $12 \times 4 = 48$

When the relationship is understood, children should be able to generalize that multiplication is used in converting from a larger unit to a smaller. It makes sense that there will be a larger number of small pieces (inches) than large pieces (feet) in a given quantity.

Counters and egg cartons may be used to develop understanding of procedure for converting from small units to larger units. Sixty counters may be dropped into compartments of the cartons and ways to determine the number of dozen of counters discussed:

The full cartons may be counted, 5

Repeated subtraction may be used,

$60 - 12 = 48$ 1 carton
$48 - 12 = 36$ 2 cartons
$36 - 12 = 24$ 3 cartons
$24 - 12 = 12$ 4 cartons
$12 - 12 = 0$ 5 cartons

Division may be used, $60 \div 12 = 5$

As this relationship is understood, children may generalize that division is used in converting from small units to large units. It is sensible that the number of large units (dozens) should be less than the number of smaller units (ones).

When manipulation of concrete objects has established understanding of the process, relationships may be expressed as ratios, and a proportion used to find the missing term in the same manner as for equivalent fractions, page 55. For example:

$$1 \text{ foot} = 12 \text{ inches}; \ \frac{1}{12} \text{ or } 1:12$$

$$4 \text{ feet} = \square \text{ inches}; \ \frac{4}{\square} \text{ or } 4:\square$$

$$\text{hence } \frac{1}{12} = \frac{4}{\square} \quad \square = 4 \times 12 = 48$$

$$12 \text{ units} = 1 \text{ dozen}; \ \frac{12}{1} \text{ or } 12:1$$

$$60 \text{ units} = \square \text{ dozen}; \ \frac{60}{\square} \text{ or } 60:\square$$

$$\text{hence } \frac{12}{1} = \frac{60}{\square} \quad \frac{60}{12} = 5$$

If ideas involved in converting from one unit to another are understood, there should be little need for manipulation of concrete materials in developing algorithms for operations on numbers used in measuring. Opportunity to experiment with examples like the following may be provided instead:

ADDITION:

$$\begin{array}{rl} & 1\text{ lb. }12\text{ oz.}\\ + & 2\text{ lb. }\ \ 5\text{ oz.}\\ \hline & 3\text{ lb. }17\text{ oz.} = (3+1)\text{ lb. }1\text{ oz.}\\ & \phantom{3\text{ lb. }17\text{ oz.}} = 4\text{ lb. }1\text{ oz.}\end{array}$$

Regrouping (17 oz. = 1 lb. 1 oz.)

SUBTRACTION:

$$\begin{array}{rll} & 4\text{ hr. }30\text{ min.} = 3\text{ hr. }(60+30)\text{ min.} & = 3\text{ hr. }90\text{ min.}\\ - & 1\text{ hr. }40\text{ min.} = 1\text{ hr. }40\text{ min.} & = \underline{1\text{ hr. }40\text{ min.}}\\ \hline & & \ 2\text{ hr. }50\text{ min.}\end{array}$$

(4 hr. = 3 hr. 60 min.)

MULTIPLICATION:

$$\begin{array}{rl} & 8\text{ yd. }\ 4\text{ ft.}\\ \times & \phantom{8\text{ yd. }\ }4\\ \hline & 32\text{ yd. }16\text{ ft.} = (32+5)\text{ yd. }1\text{ ft.}\\ & \phantom{32\text{ yd. }16\text{ ft.}} = 37\text{ yd. }1\text{ ft.}\end{array}$$

Regrouping (16 ft. = 5 yd. 1 ft.)

DIVISION:

$$\begin{array}{rl} & 1\text{ yr. }7\text{ mo.}\\ \cline{2-2} 4\,\big) & 6\text{ yr. }4\text{ mo.}\\ & 4\text{ yr.}\\ \cline{2-2} & 2\text{ yr. }4\text{ mo.} = 28\text{ mo.}\\ & \phantom{2\text{ yr. }4\text{ mo.} = }\ \underline{28\text{ mo.}}\end{array}$$

Regrouping (2 yr. = 24 mo.)

Estimation of Measurement

Intuitive understanding of measures may be developed through practice in estimating length, weight, height, or temperature, then measuring to check estimates. Children may step off distances, then measure length of each stride to form a basis for estimating distance. They may use their own height and weight as basis for estimating height and weight of others. Objects weighing one ounce, 8 ounces, and one pound may be handled to develop an intuitive idea for estimating weight. Sets of objects may be used to develop skill in estimating the number of discrete objects: number of seats in auditorium, number of students in the cafeteria, number of windows in a building, number of beans in a pound.

Perimeter

Understanding of perimeter as distance along a simple closed curve may be developed by providing opportunity for children to walk completely around many kinds of simple closed paths (painted lines on tennis, basketball, or shuffleboard courts) counting steps as they go. Perimeter thus becomes associated with distance to be measured.

The derivation of the word perimeter from Greek words for "around" and "measure" may be used to strengthen ideas of perimeter.

Polygon: As simple closed paths are classified by shapes, polygons may be constructed from paper, strips of cardboard, pipe cleaners, string on pegboard, bent coat hangers, or other materials. As the perimeter of each type is measured, simple formulae may be written and recorded in chart form. For example:

regular hexagon—

$P = s + s + s + s + s + s$
$P = 6 \times s$

rectangle or parallelogram—

$P = a + b + a + b$
$P = 2a + 2b$
$P = 2\ (a + b)$

triangles—

$P = s + s + s$ or
$P = a + a + b$ or
$P = a + b + c$

Circle: The perimeter of a circle is designated by a special name, circumference, which may be measured by using another measurement, the diameter. The relationship of these two measurements may be demonstrated by using string and a cardboard model of a circle. A detailed description of the manner in which this is done can be found in Measurement, page 132.

This procedure may be repeated using many circles of different size, demonstrating that in each instance the circumference is approximately three times the length of the diameter. When children become familiar with operations on decimal fractions, a more precise quotient, 3.14^{+} designated by the term pi or the Greek letter π, may be derived.

Since the circumference is approximately three times the length of the diameter, a formula for finding the circumference of any circle may then be developed by using the diameter, $C \approx 3.14^{+} \times D$ or $C = \pi \times D$. Again, effort should not be made to have children memorize the formula but rather to be able to demonstrate a means of determining the circumference of any circle if its diameter can be measured.

Area

Understanding of area may be developed through activities that apply unit regions of various sizes and shapes to flat surfaces. Desk tops or other rectangular regions may be covered with sheets of paper, rectangular unit regions, without overlapping. As the activity is extended to include triangular regions to be measured, it will become evident that triangular shaped units are more appropriate for covering these regions without overlapping.

Discussion of situations like the following may help in development of understanding of the relationship of perimeter and area:

1. John's father is building a fence around the yard. Must he know the area or perimeter of the yard?
2. Betsy's father is covering the tomato patch with mulch paper. Must he know the area or perimeter of the patch?
3. Jack is painting the dog house. Must he know the area or perimeter before buying paint?

1. AREA OF POLYGONS

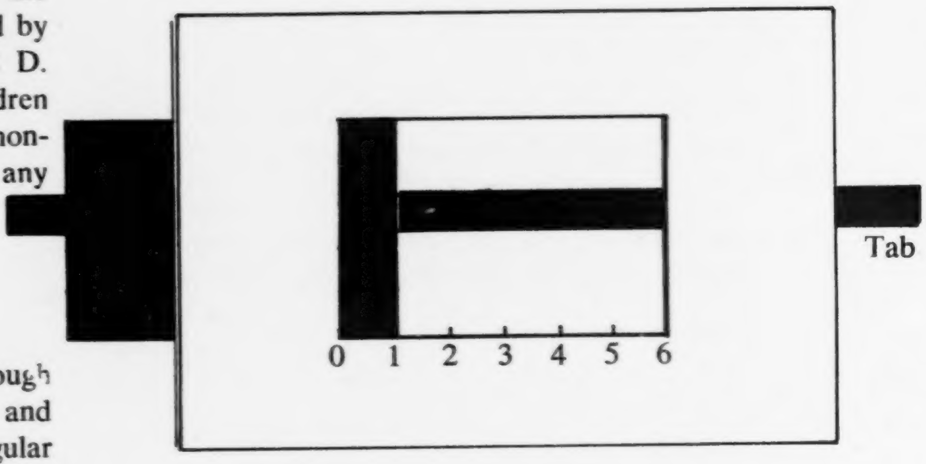
To develop generalizations or formulae for finding areas of various polygons such activities as the following may be used:

Rectangular regions:

a. Applying unit regions to various sized rectangular regions and recording the process by listing the number of units per row, the number of rows of units and the total number of units used.

b. Counting in a given area of floor space the number of tiles in a row, the number of rows of tiles and the total number of tiles used.

c. Constructing a revealer as illustrated below to show row by row the measuring process.



As the tab is pulled, a row of unit regions appears to illustrate 1 row of 5 units. As the action is continued, successive rows as marked along the horizontal dimension of the region being measured are reached. 1 row of 5 units (1×5), 2 rows of 5 units (2×5) . . . until the total region is covered, 6 rows of 5 units (6×5) or 30 units of measure.

d. Discussing the process as used in measuring rectangular spaces and stating the generalization that the measure of the area of a rectangle $= l \times w$ or $A = b \times h$.

Parallelogram region:

a. Making paper models of parallelogram regions.

b. Cutting parallelogram and reconstructing figure to form a rectangle as shown in illustration on page 133.

c. Discussing the resulting figure and stating the generalization that the measure of the area of parallelogram is length times width. $A = l \times w$ or $A = b \times h$.

Trapezoidal region:

a. Making models of two congruent trapezoidal regions.

b. Suggesting that children arrange the two figures to produce a third figure with which they are familiar. (See illustration on page 134.)

c. Discussing the resulting figures and stating the generalization that the area of a trapezoidal region is equal to one half of the sum of the measure of the two bases times the measure of the height. $A = \frac{1}{2}(b_1 + b_2) \times h$.

Triangular region:

a. Making models of two congruent triangular regions.

b. Suggesting that children arrange these to form a familiar figure. (See illustration on page 134.)

c. Discussing the resulting figures and stating the generalization that the area of a triangular region is equal to one half of the area of a parallelogram region. $A = \frac{1}{2} b \times h$.

Circular region:

a. Making models of a circular region and of a square with side measures twice the length of the radius of the circle; applying circle to square as in diagram on page 134 to illustrate the relationship and to develop the generalization that the area of the circle is slightly less than the area of the square.

b. Making model of a circular region:

(1) Cutting it similar to illustration on page 135

(2) Fitting sections together to form an approximation of a parallelogram region

(3) Repeating the process in 2 above using circles with radii of the same measure, but cut into smaller and smaller sections

(4) Discussing that the smaller the size of the section, the closer the approximation to the size of a parallelogram which has a base approximately one half the circumference of the circle and height equal to the measurement of the radius of the circle.

c. Stating the generalization that the area of a circle is equal to the radius times $\frac{1}{2}$ the circumference. The circumference is equal to $2\pi r$.

$$A = \frac{1}{2}(2\pi r) \times r$$

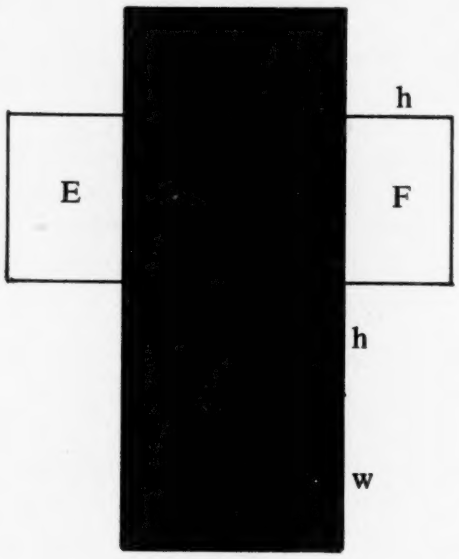
$$A = \pi r \times r$$

$$A = \pi r^2$$

2. AREA OF CLOSED SURFACES

To develop formulae for finding the areas of closed surfaces, such activities as the following may be used:

Prism:



a. Opening corner seams of a large cardboard carton and spreading it flat as in illustration.

b. Measuring dimensions of each section of the box

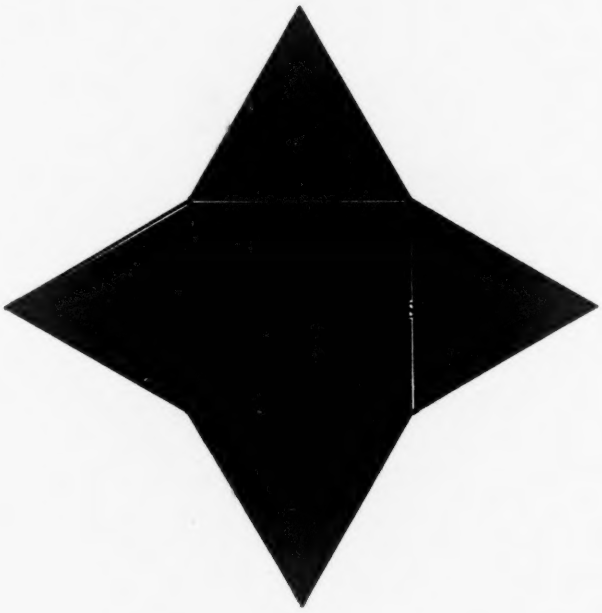
c. Computing the area of each plane surface and developing a formula.

Area of $A = l \times w$
Area of $B = l \times w$
Area of $C = h \times l$
Area of $D = h \times l$
Area of $E = h \times w$
Area of $F = h \times w$

$A = B$
$C = D$
$E = F$

Total surface area $= A + B + C + D + E + F$
$= 2\,(l \times w) + 2\,(h \times l) + 2\,(h \times w)$ or
$= 2B + h \times 2\,(l + w)$ or
$2B + (h \times p)$ where $B =$ area of the bases and
$p =$ perimeter of the base.

Pyramid: Drawing flattened models of a regular pyramid with a square base as illustrated to the right and proceeding as for rectangular prism to develop formulae for finding the area of the closed surface.



Area of $A = s \times s = s^2$

Area of $B = \frac{1}{2}\,(s \times h)$

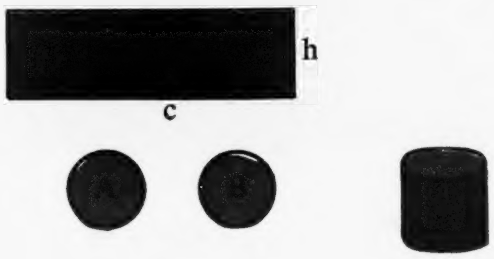
Area of $C = \frac{1}{2}\,(s \times h)$

Area of $D = \frac{1}{2}\,(s \times h)$

Area of $E = \frac{1}{2}\,(s \times h)$

Area of closed surface of regular pyramid $= B + \frac{1}{2}\,(h \times p)$ where $B =$ area of base and $p =$ perimeter of base.

Cylinder:



a. Fitting paper around a can or circular box and cutting circles to match ends of cylinder

b. Spreading paper and measuring length and width of rectangle and radius of circles

c. Computing area of plane surfaces and developing formula.

Area of $R = c \times h$

Area of $A = \pi \times r^2$ $\qquad A = B$

Area of $B = \pi \times r^2$

Total surface area $= R + A + B$
$= (c \times h) + 2\,(\pi r^2)$ where
$c =$ circumference of base of cylinder

Volume

Understanding of volume may be developed through activities that fit unit space regions into space regions of various sizes and shapes. Blocks may be fitted into a box, or empty spools into a paper tube of the same diameter.

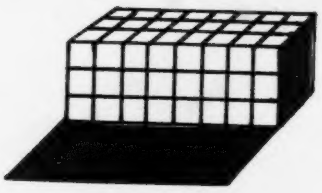
If spools are fitted into a rectangular box, it should become evident that space regions of appropriate nature must be used for accurate measurement. As understanding of volume as measure of interior space is established, the need for use of standard units should also be clear.

Discussion of such situations as the following may aid in further clarification of understanding of volume:

1. Tom is ordering sand for his little sister's sand box. Must he know volume, perimeter, or area?

2. In estimating water needed to fill the aquarium, must the class know perimeter, volume, or area?
3. In estimating the amount of water in the water tank, must the city engineer know volume, perimeter, or surface area?

Rectangular prism: As the nature of units needed to measure volume of rectangular prisms is discussed, cubes of different sizes may be examined and dimensions for standard units discussed. 1 inch and 12 inch cubes may be constructed from heavy cardboard.



The bottom of a rectangular pasteboard box may be covered with unit regions to make a layer to cover the area of the base, and other layers added until the box is filled. If the box is slit along the edges of one side as shown in the illustration, the layers can be revealed. It can be seen that the volume of the rectangular box or prism is equal to the area of the bottom layer times the number of layers or

Volume of prism = $(l \times w) \times h$ or
= $B \times h$ where B = area of base.

Regular pyramid: When procedure for finding the volume of a prism is understood, the idea may be used to determine the volume of a pyramid. Construction illustrated on page 137 may be used to develop understanding of the relationship of prism and pyramid with bases of the same measure. Models of prism and pyramids may be constructed. It can be demonstrated that the model of the pyramid filled three times will fill the prism. It should be evident that the volume of a pyramid is equal to one third of the volume of a prism with base and altitude of the same measures or

Volume of a regular pyramid = $\frac{1}{3}B \times h$ where B represents the area of the base.

Right circular cylinder: By adapting ideas used in finding the volume of rectangular prisms, children should be able to visualize the volume of a cylinder. Since the base is a circular region, the volume of the cylinder will be equal to the area of a circle times the measurement of the height of the cylinder. $V = \pi r^2 \times h$

Right circular cone: When procedure for finding the volume of a cylinder is understood, the idea may be used to determine the volume of a cone by:

a. Constructing a cone as illustrated on page 124
b. Constructing a cylinder with the height and diameter of the cone
c. Using the cone to fill the cylinder.

It will be apparent that the volume of a cone is one third the volume of the cylinder. $V = \frac{1}{3}\pi r^2 \times h$

PROBLEM SOLVING

As stated in the "Point of View," problem solving is as much an *approach to teaching* application of mathematical content as it is the process of finding answers to word or verbal problems. However, the solution of word problems remains as a major goal at each grade level of elementary mathematics. Solving word problems with applications to science and other subject fields is of increasing importance. This makes it imperative that careful and systematic attention be given to problems in the mathematics program.

Because problem solving is an important goal and because teachers often encounter difficulty in teaching verbal problems, a special section of this guide is provided.

The essence of solving word problems may be summarized as follows:

A. The words of verbal problems suggest some kind of action—either physical movement or an imagined operation.

B. A mathematical model is set up to fit this action or imagined operation. It is at this stage that the decision is made on the mathematics needed to fit the problem.

C. The mathematical problem is solved.

D. The mathematical solution is then interpreted back into the problem situation described by the words.

Example: At the equator, the distance around the earth is about 25,000 miles. About how many miles would you travel if you went around the earth twice along the equator?

A. The action described is "travel twice around the equator."

B. The learner begins to search his repertoire of mathematical models. He may think: "Around once; around twice; 25,000 miles each time" He has chosen an addition model that fits the words.

The learner might think: "Around once; around twice; two times I went 25,000 miles." He may write: $2 \times 25{,}000 = n$. He has chosen a multiplication model that fits the word description.

C. Solving the mathematical problem:

$$25{,}000 + 25{,}000 = n \qquad 2 \times 25{,}000 = n$$

$$50{,}000 = n \qquad 50{,}000 = n$$

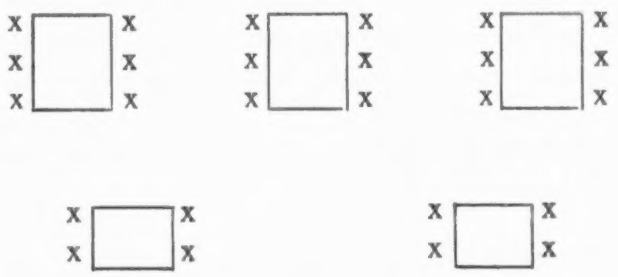
D. The solution to the mathematical model is 50,000. Interpreting this back into the physical situation is shown by the sentence: I would travel about 50,000 miles if I went around the earth twice along the equator.

Illustrated below are some practical suggestions that teachers have found helpful. Most of these suggestions are designed to help with aspects of problem solving described in A or B above.

1. Children may make problems to fit a given number sentence. This can help them to see the wide variety of situations that give rise to a single mathematical model and it insures problems that relate to life experiences of the children involved.

   Example: $5 + 3 = \square$ may become a number story about boys and girls at a party, apples and pears in a fruit bowl, or kittens and puppies at a pet show.

2. Diagrams may be drawn to assist in formulation of the mathematical model.



   Example: There are 28 pupils in the third grade. In the room there are 3 tables for 6 children each and 2 tables for 4 children each. Are more tables needed to give each pupil a space at a table?

   $6 + 6 + 6 + 4 + 4 = 26$

   $(3 \times 6) + (2 \times 4) < 28$ Yes, 1 more table is needed to seat 2 children.

3. The problem may be dramatized to help pupils to visualize the action of the problem.

4. Children may dictate or write their own problems, write appropriate number sentences, and then solve the sentences. This assures problems that are understood and within the difficulty level, both language and mathematics, of each student.

5. Teachers may make problems involving situations of current interest to the class.

6. Information given in the problem may be summarized to emphasize the action suggested.

   Example: Helen and Clara are stacking programs for the class play into sets of 20. If they have 180 programs, how many stacks should they have?

   As possible summaries children may either

   a. Draw pictures of the action
   b. Put 180 counters into sets of 20
   c. Ask themselves questions:

      180 programs make what number of stacks of 20?

      How many stacks of 20 are needed to make 180?

   From the summary, a mathematical model is easier to formulate into a number sentence:

   $180 = \square \times 20$

   $\square \times 20 = 180$

   $180 \div 20 = \square$

   Finding the number that will make the equation true will provide solution to the problem.

7. The problem may be read, an estimation of the answer made, and the problem reread to check the reasonableness of the answer. Such activity helps in the formulation of the mathematical model to be used.

   Example: If Jack's father travels an average of 50 miles each hour on a trip, how many hours does it take him to drive 375 miles? Estimates might be made by thinking:

   100 miles —2 hours; 300 miles—6 hours

   375 miles almost 8 hours

   When the problem is reread and the estimation inserted, reasonableness may be discussed and checked by children's experiences with travel.

8. Problems containing information not necessary to the solution may be used to help children identify the true problem situation.

   Example: $\frac{1}{2}$ of the 150 pieces of mail delivered by the postman in one city block were first class mail and $\frac{1}{3}$ were second class. How many pieces of second class mail were delivered?

9. If problems dealing with large numbers, fractions, or decimals present difficulties, they may be approached by:

   a. Substituting small whole numbers for numbers in the problem
   b. Reading and solving the problem using the whole numbers
   c. Discussing the mathematical model used
   d. Using the same operation with the original numbers.

   Example: The class had $15\frac{2}{3}$ dozen cookies for a bake sale. If $1\frac{1}{2}$ dozen were left at the end of the sale, how many were sold?

   a. The class had 15 dozen cookies for a bake sale. If 1 dozen were left at the end of the sale, how many were sold?
   b. Think: $15 - 1 = n$ or $1 + n = 15$
   c. Solve:
   d. Think: $15\frac{2}{3} - 1\frac{1}{2} = n$ or $1\frac{1}{2} + n = 15\frac{2}{3}$

10. Several number sentences related to a given problem may be written, children asked to select those which may be used in the solution of the problem, and appropriateness of selections discussed.

    Example: Jack earned 45¢ each time he cleared snow from the front walk. If he shoveled snow four times in January, how much did he earn that month?

    a. $45 + 45 + 45 + 45 = n$
    b. $45 \times n = 4$
    c. $n = 45 \times 4$
    d. $45 \div 4 = n$

11. Number sentences may be written for problems without attempting to solve them in order to focus attention on developing a mathematical model that fits the problem.

    Example: The grocer charges 70¢ a dozen for large eggs and 60¢ a dozen for small eggs. How much does he get for 1 dozen large eggs and 3 dozen small eggs?

    Physical models may be drawn, then mathematical models stated:

$70 + (3 \times 60) = n$

$(3 \times 60) + 70 = n$

12. Problem situations described without using numbers may be used to focus attention on mathematical operations to be used.

Example: If each member of a class contributes the same amount to a fund for a class party, how much money will there be to spend?

Questions similar to the following may be asked to guide thinking:

a. What do you know from reading the problem?

b. What are you asked to find?

c. Can you write a sentence that tells what can be done to find the answer? (Number of people multiplied by amount of money given by each person is the amount of money to spend.)

d. If you insert numbers in the story, is the answer reasonable? (25 is the number of pupils who contributed; 20¢ is the amount contributed by each pupil; n or ☐ is the amount of money for the party.

$25 \times 20 = n$

$500 = n$ There will be 500 cents or $5.00 to spend.)

As a means of introducing new mathematical topics, a practical problem may be presented for which students do not have an appropriate mathematical model or for which they may need a more efficient model. Such problems provide motivation for learning the topic and may give the learner a sense of realism in the study of the topic.

For example: As one way to introduce ratio and proportion, the problem of increasing the number of cup cakes produced by a recipe may be presented. If the recipe makes 2 dozen and 6 dozen are needed for a party, the relationship involved is 6 to 2 or a ratio of $\frac{3}{1}$. The recipe may be increased in the following manner:

Recipe for 2 dozen cup cakes	Proportion for 6 dozen cup cakes
1 c butter	$\frac{2}{1} = \frac{6}{\square}$ = 3 c butter
2 c sugar	$\frac{2}{2} = \frac{6}{\square}$ = 6 c sugar
4 eggs	$\frac{2}{4} = \frac{6}{\square}$ = 12 eggs
1 c milk	$\frac{2}{1} = \frac{6}{\square}$ = 3 c milk
3 c flour	$\frac{2}{3} = \frac{6}{\square}$ = 9 c flour
2 t baking powder	$\frac{2}{2} = \frac{6}{\square}$ = 6 t baking powder
$\frac{1}{2}$ t salt	$\frac{2}{\frac{1}{2}} = \frac{6}{\square}$ = $1\frac{1}{2}$ t salt
2 t flavoring	$\frac{2}{2} = \frac{6}{\square}$ = 6 t flavoring

Practical problems may be interwoven with mathematical topics being studied, particularly when the mathematical topic loses a sense of reality.

For example: During the study of percent the federal government announces an increase in the interest rate on Type E savings bonds. This may be used not only to give reality to computation of interest but can lead to greater understanding of application of per cent in life beyond the classroom. There should be continuing interplay between mathematics studied and verbal or practical problems.

Certain actions occur often in practical problems. That actions relate to the four operations on numbers is not accidental. Summarized below are actions that lead to the formulation of given operations with whole numbers.

Action	Problem	Mathematical Model
Joining or combining two sets with numbers of each known	How many in the resulting set?	Addition: $5 + 17 = n$
Separating a set into two subsets with numbers of original set and one subset known	How many in the other subset?	Subtraction: $17 - 5 = n$ or $5 + n = 17$
Comparing one set with another with numbers of each known	How many more in one set than in the other? How many fewer in one set than the other?	Subtraction: $17 - 5 = n$ or $5 + n = 17$
Joining equivalent sets with the number of sets and the members of each set known	How many in the resulting set?	Multiplication: $5 \times 17 = n$ or Addition: $17 + 17 + 17 + 17 + 17 = n$
Separating a set into equivalent subsets with the number of the original set and of each subset known	How many sets?	Division: $35 \div 5 = n$ $n \times 5 = 35$ zero remainder
	How many sets and how many left over?	$35 = n \times 5$ or $37 = (n \times 5) + r$ non-zero remainder
Separating a set into equivalent subsets with the number of the original set and the number of subsets known	How many in each set?	$35 \div 5 = n$ $5 \times n = 35$ zero remainder
	How many in each set and how many left over?	$35 = 5 \times n$ $38 = (5 \times n) + r$ non-zero remainder

Pupils must learn that to read word problems in mathematics demands reading skill that differs from that used in other subject areas. Problems must be read carefully several times, each time for a different reason. They must be read to:

1. Get overview of the problem situation.
2. Garner necessary facts.
3. Determine mathematical model.
4. Check the reasonableness of the solution of the model.
5. Interpret the solution in words of the problem.

Children must realize that each problem is different and that since no set pattern can be developed for solving all word problems, each must be carefully read.

The assignment of pages of word problems for pupils to do on their own is certain to produce failure for many. Alone they are unable to formulate the mathematical model, the crucial step in problem solving. Many pupils proceed by trial and error using each operation until they get a number that "looks about right." When this happens, little has been achieved with the major goal of problem solving and a negative attitude is often acquired. Pupils who are successful usually have experienced a carefully planned, sequentially developed program of instruction that matched individual understanding with appropriate application. It is generally recognized that it is better for a pupil to do fewer problems successfully than to attempt many and solve few.

STATEMENT OF OBJECTIVES

INTRODUCTION

The Statement of Objectives outlines mathematical knowledge and skills to be emphasized during the first eight years, grades K-7, of the elementary school. This Statement was designed as a guide in planning for sequential development of basic mathematical ideas. Objectives should assist teachers:

In assessing competencies which each pupil brings to a given grade

In planning experiences needed for further growth in mathematical understanding and skill

In developing understanding of more advanced mathematics which children will encounter.

Effort has been made to express objectives in terms of what children might be able to do after a year's experience in a given grade. The Statement is not intended to provide a set of minimum essentials for a child at any grade level.

Grade placement of content as outlined is suggestive. Objectives may need to be restated in terms of conditions existing in individual schools. Special attention should be given to objectives for primary grades in schools where kindergarten is not a part of the system. Grade placement of content must be interpreted and used by every teacher in light of what individual children know and are able to do at a given time.

A characteristic of the study of mathematics is the spiraling of concepts from grade to grade over a period of years. For example, multiplication begins in the first grade with the manipulation of equivalent sets of concrete objects, continues through the set of whole numbers and expands to include the set of fractional numbers and the set of rational numbers.

Organization of the Guide was determined by the belief that certain broad basic ideas unify mathematical content. Six major strands—Sets and Numbers, Numeration, Operations on Whole Numbers, Rational Numbers, Geometry, and Measurement—have been selected as having greatest significance in mathematics for the elementary school. Effort has been made to state objectives in a way that suggests interrelatedness of these ideas in a spiraling development of content.

Because mathematical ideas presented in the six strands are closely interrelated, it is possible that an idea may develop spirally through more than one strand. For example, manipulation of concrete and semi-concrete materials basic to understanding operations on whole numbers, may appear in objectives for Sets and Numbers and Numeration. This should be kept in mind as this section of the Guide is used.

The manner in which mathematical ideas are introduced and developed in the Guide should be carefully noted. Understanding which children may have gained in pre-school years is used in expanding ideas of number and space. Work with concrete and semi-concrete materials develops these ideas in greater depth before symbols are used, generalizations stated, or operations with numbers formalized.

Mathematical content must be organized and presented in a sequence which enables each child to develop understanding of basic ideas. Each teacher must be knowledgeable concerning content presented at all levels in order to insure maintenance of skills and understanding and to plan next steps in learning. No gaps may be allowed in gradual build up and expansion if the school is to provide a complete program for each child.

OBJECTIVES

KINDERGARTEN	FIRST GRADE
Sets and Numbers	
Identify two equivalent sets by placing the members of the sets in one-to-one correspondence.	Recognize that the equality symbol means that the same number is named on either side of the symbol.
Recognize the common property of all sets as number.	Use 0 as the cardinal number of the empty set.
Name the number of members in sets with 1-9 members by counting.	Order sets of objects with numbers to ten.
	Order numbers using concept of “one more than,” and “one less than,” and “equal to.”
	Recognize and use number words for ordinals through five. (number five and fifth)
	Count to 100 by ones and tens, to 20 by twos, to 50 by fives.
Select 1-9 objects from a given set using oral words and written numerals.	Recognize and use number words through 10.
Compare sets of objects using such terms as *more than, as many as, fewer than.*	Compare two numbers using “is greater than,” “is less than,” or “is equal to” and use symbols “>,” “<,” or “=.”
	Recognize and demonstrate one half and one fourth of a set of objects.
	Join two sets to make a third set; separate a given set into two sets.
Numeration	
Recognize and name the symbols 1 through 9.	Group sets of objects using the base of ten and name the number as so many sets of ten and so many ones.
	Use place value notation for numbers through 99 to show the number of tens and ones.
	Read and write numerals 0 through 99.
	Read and write number words to ten.
	Recognize that a number has many names. (3 = 2 + 1 = 3 + 0 = 4 − 1)

SECOND GRADE	THIRD GRADE

Sets and Numbers

SECOND GRADE	THIRD GRADE
Know odd and even numbers.	
Use ordinal numbers through ten. (Number ten and tenth)	Use ordinal numbers beyond ten.
Count to 1000 by tens and hundreds.	Use whole numbers to ten thousand.
Recognize and demonstrate one half, one fourth, one third, using sets of objects.	Recognize and demonstrate one half, one fourth, three fourths, one third, and two thirds, using sets of objects.

Numeration

SECOND GRADE	THIRD GRADE
Read and write three-place numerals using place value notation to show the number of hundreds, tens, and ones.	Read and write four-place numerals using place value, expanded notation, and word names.
Count, read, and write numerals from 0 to 300.	Read and write decimal notations for money.
Read and write many names for numbers. Example: $5 + 3 = 8$; $10 - 2 = 8$; $4 + 4 = 8$.	
	Read and write Roman numerals through X.

KINDERGARTEN	FIRST GRADE
Operations on Whole Numbers	
Rearrange sets of objects to demonstrate the joining and separating of sets to develop a readiness for addition and subtraction.	Know and use addition and subtraction facts for sums through 10.
	Explore the facts with sums through 18.
	Recognize the commutative property of addition without naming it.
	Add using three addends with sums through ten.
	Use intuitively the associative property of addition.
	Use vertical and horizontal forms for addition.
	Use the symbols +, −, and = to form number sentences. Example: 3 + 6 = □; 6 − 3 = □
	Explore addition of numbers named by two digits without renaming.
	Demonstrate with sets of objects the inverse relationship between addition and subtraction. 4 + 2 = 6; 6 − 2 = 4; 6 − 4 = 2
	Use multiplication and division intuitively without symbolization with products to ten.

SECOND GRADE	THIRD GRADE
Operation on Whole Numbers	
Know and use addition and subtraction facts with sums through 18.	
Recognize that subtraction is not commutative.	
Recognize and use the associative property of addition without naming it. Example: 7 + (3 + 5) = (7 + 3) + 5.	
Add and subtract numbers named by two digits without renaming.	Add and subtract numbers each named by two or three digits without renaming.
Add numbers named by two digits, renaming as necessary.	Add numbers named by two and three digits with renaming.
Use the inverse relationship of subtraction and addition to derive subtraction facts from the addition facts.	Subtract numbers each named with two or three digits with renaming.
Check subtraction by addition.	
Use odd and even numbers in developing and checking addition and subtraction.	Use odd and even numbers in developing ideas of multiplication and division and checking products and quotients.
Demonstrate multiplication through joining sets of objects, repeated addition, and rectangular arrays.	
Know and use multiplication facts with products through ten.	Know and use multiplication facts with one factor 0-6, and the other factor 0-9. Explore facts with both factors 7-9.
	Use multiplication with one factor a number named by one digit and the other factor a number named by either two digits or three digits.

KINDERGARTEN	FIRST GRADE
	Recognize the number of 2's, 3's, 4's, and 5's in multiples of these numbers to 10.
	Use numbers with understanding in stories, games, and related activities.
Rational Numbers	
Recognize that single unit objects can be split into two equal pieces, and that each piece is called one half of the object.	Recognize and demonstrate $\frac{1}{2}$ and $\frac{1}{4}$ of physical units and of sets of objects.
Recognize informally that sets of 2, 4, 6, and 8 objects can be separated into two equivalent subsets as readiness for understanding that each subset is one half of the original set.	Assign the number name $\frac{1}{2}$ to each of two equal sized pieces of a physical unit and to each of two equivalent subsets of a set of objects.

SECOND GRADE	THIRD GRADE
	Know and use the properties of zero and one in multiplication.
	Recognize the role of the distributive property of multiplication over addition and its relationship to the algorithm. Example: $3 \times (2 + 4) = (3 \times 2) + (3 \times 4) = 6 + 12 = 18$ $4 \times 23 = 4 \times (20 + 3) = (4 \times 20) + (4 \times 3) = 80 + 12 = 92$ $\begin{array}{r} 23 \\ \times 4 \\ \hline 12 \\ 80 \\ \hline 92 \end{array} \qquad \begin{array}{r} 23 \\ \times 4 \\ \hline 92 \end{array}$
	Recognize inverse relationship of multiplication and division through use of multiplication facts.
Recognize that multiplication is commutative.	Recognize that division is not commutative.
Demonstrate division through partitioning sets of objects, repeated subtraction, and rectangular arrays.	Use long division algorithm with divisor a number named by one digit (1-6) with and without remainders.
Apply addition and subtraction to problems involving money using cents only.	Apply addition and subtraction to problems involving money, including dollars and cents.
Rational Numbers	
Recognize and demonstrate $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ of a physical unit, a set of objects, a region, and a line segment.	Recognize the order relationship (greater than, less than) for fractions of eighths, sixths, fifths, fourths, thirds, and halves.
	Recognize that there is a point on the number line for every fractional number.
	Recognize that there are many fractions that name the same fractional number.
	Recognize the inverse relationship of addition and subtraction of fractional numbers.
Find one half of each even number from 2 through 10.	Find one half of each even number from 2 through 20.
	Recognize that a fraction names an ordered pair of numbers, as well as a fractional number.

KINDERGARTEN	FIRST GRADE
Geometry	
Recognize and name squares, rectangles, circles, and triangles.	Recognize that squares, rectangles, triangles, and circles are closed paths. Tell whether a given point is inside, outside, or on such a path.
Relate terms *inside, outside,* and *on* as they apply to squares, rectangles, circles, and triangles.	Recognize some distinguishing features of circles, rectangles, triangles, and squares. Use some related terms such as round corner, square corner, paths that are curved, and paths that are straight.
Recognize physical models and drawings of points, line segments, and portions of a plane (flat surface).	Recognize and name a line with the understanding that it means a straight path and that it extends indefinitely in both directions.
Identify objects shaped like a ball, box, or can.	Identify objects shaped like a ball, a box, a can, an egg, a cone.
Measurement	
Compare objects through use of such words as longer-shorter, heavier-lighter, higher-lower, larger-smaller.	Use non-standard units of linear measure and liquid measure, such as a pencil for length and paper cup for liquid measure, to illustrate the necessity for standard units.

SECOND GRADE	THIRD GRADE
	Interpret simple ratio situations, such as 3 pencils for 10 cents, as an ordered pair or fraction.
Geometry	
Distinguish between open and closed curves and inside and outside of simple closed curves.	
Recognize and use properties of simple geometric figures such as square corners, opposite sides, and roundness.	Recognize basic properties of triangles, rectangles, squares, and circles without use of formal definitions.
	Recognize that a rectangular sheet of paper can be folded into two or more parts of the same size and shape.
Recognize and use the words: point, line segment, and line to describe geometric ideas.	Recognize a point as a location in space as distinct from its physical representation.
	Recognize a line as a set of points that continues on indefinitely.
	Recognize a line segment as composed of the two endpoints and all the points between them.
	Recognize that there is only one line through two points.
	Recognize that two different lines can intersect at only one point.
	Recognize and name rays and angles.
	Recognize symmetry with respect to a line by using such activities as simple paper folding.
Recognize geometric figures and objects which have the same size and shape.	Recognize without formal terminology distinguishing features of spheres, rectangular prisms (boxes), and cylinders.
Measurement	
Recognize proper instruments for measuring different things, such as ruler, clock, cup, calendar, and scales, as readiness for selecting units of measure.	Recognize the necessity for standard units of measure.
	Recognize that the measure of a line segment is the number obtained by comparing the line segment to a selected unit line segment.

KINDERGARTEN	FIRST GRADE
	Determine length of line segments to the nearest inch using a ruler marked only in inches.
	Explore liquid measures using cups, pints, and quarts.
	Read and use a calendar for days and weeks.
	Tell time to the nearest half-hour.
Identify pennies, nickels, and dimes.	Recognize the compatative values of pennies, nickels, and dimes and use them in making change.
	Recognize that there is a point on a line for each whole number.

SECOND GRADE	THIRD GRADE
Use a ruler marked in half-inch, inch, and foot units to measure line segments.	Use standard units: inches, feet, and yards in measuring line segments; cups, pints, quarts, and gallons in measuring liquids.
Measure liquids by cups, pints, quarts, and gallons.	
Read and use a calendar for days, weeks, and months.	Use year as measure of time.
Tell time to the nearest quarter-hour.	Tell time to the nearest minute and use notation such as 3:15 P.M.
Measure weight in pounds.	Use ounces to measure weight.
	Measure temperature on the Fahrenheit scale to nearest degree.
	Find the perimeter of triangles and rectangles by adding the measures of the sides.
	Recognize that a measurement of physical objects is always an approximate number.
Make change up to a quarter.	Make change up to $1.00.
	Read and interpret simple information given in tables.
	Recognize that there is a point on a line for each fractional number.

OBJECTIVES

FOURTH GRADE	FIFTH GRADE
Sets and Numbers	
Recognize and use sets of numbers, including counting numbers, whole numbers, even numbers, and odd numbers.	Identify the set of prime numbers 2, 3, 5, 7, 11, . . .
	Find the prime factors of counting numbers through 100.
	Identify the set of composite numbers 4, 6, 8, 9, 10, 12, . . .
	Determine the least common multiple of two counting numbers by listing the multiples of the two numbers and by prime factorization.
	Determine the greatest common factor of the counting numbers by listing factors of both numbers and by prime factorization.
Recognize sets of fractional numbers and equivalent fractional numbers.	Identify the set of fractional numbers.
	Construct sets of equivalent fractions.
	Use set notation of union and intersection.
Numeration	
Read and write seven place numerals as needed.	Read and write decimals utilizing extension of place value to two places to the right of the one's place.
	Use exponents ^{2}, ^{3}, or ^{4}. Example: $10^2 = 100$; $3^3 = 27$; $4000 = 4 \times 1000$ $= 4 \times 10 \times 10 \times 10$ $= 4 \times 10^3$
Use Roman numerals through one hundred with emphasis on base of ten and the use of a subtractive principle in a numeration system. Examples: Ten ones make one X, ten tens make one C, ten hun-	Read and write Roman numerals.

SIXTH GRADE	SEVENTH GRADE
Sets and Numbers	
Use set of negative integers in practical situations such as naming temperatures above and below zero.	Use the entire set of rational numbers.
	Recognize that there are numbers called irrational numbers that are not rational.
Use solution sets in equations and inequalities.	
Use fractional numbers including decimal representation.	
	Use set terminology in referring to sets of points in geometry.
Numeration	
	Understand the role of place value in a numeration system through the study of non-positional systems such as the Egyptian.
Use expanded notation with exponents. Example: $4526 = (4 \times 10^3) + (5 \times 10^2) + (2 \times 10^1) + (6 \times 10^0)$	Express numbers by using scientific notations. Example: The distance from the earth to the sun, 93,000,000 miles, is 9.3×10^7 miles.
Represent fractional numbers by decimals and fractions.	

FOURTH GRADE	FIFTH GRADE
dreds make one M, IV = one less than V and XL = ten less than fifty.	
	Read and write numerals in a base other than 10.
Operations on Whole Numbers	
Add and subtract numbers named by four or more digit numerals without renaming.	
Rename as necessary when using numbers named by four or more digits for addition and subtraction.	
Add and subtract numbers applied to various types of measures without renaming.	Add and subtract numbers from measurement and money, renaming as needed.
Know and use multiplication facts with factors 0 through 10.	
Multiply with one factor a number named by two digits and the other factor a number named by three digits.	Use multiplication with factors named by three and four digits.
Use the terms associative and commutative properties of addition and multiplication after understanding the ideas.	
Use the term distributive property after understanding the idea.	
Use division algorithm with divisors named by two digits ending in 1, 2, 3, or 4 with and without remainders.	Use division with divisor named by two digits.
Rational Numbers	
Determine order relationship for fractional numbers.	
Find an equivalent fraction for a given fraction through use of models.	
Recognize that fractions such as $\frac{1}{1}, \frac{2}{2}, \frac{3}{3} \ldots$, are names for one.	
	Know and use decimals to thousandths.

SIXTH GRADE	SEVENTH GRADE
Operations on Whole Numbers	
Use simple addition and subtraction in bases other than ten.	
Multiply and divide with numbers from measurement and money, renaming when needed.	Use simple multiplication in bases other than ten.
Use division with divisors named by three digits.	Use division with divisors named by four or more digits.
	Find square roots of numbers represented by two and three-digit numerals.
Rational Numbers	
	Use order relations =, >, < with integers and rational numbers.
Recognize that there is a fractional number between every two fractional numbers on the number line.	Recognize the idea that the integers are not dense, but that rational numbers are dense.
Demonstrate an understanding of decimals as names for certain fractional numbers.	Recognize the additive inverse property that every fractional number has an opposite.
Know decimal equivalent for fractions such as $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{5}$, $\frac{1}{8}$.	

FOURTH GRADE	FIFTH GRADE
Add and subtract fractional numbers with same denominator.	Add and subtract fractional numbers with unlike denominators.
	Add and subtract decimals to thousandths.
	Recognize that subtraction is not commutative in the set of fractional numbers.
	Know and use the commutative and associative properties for addition in the set of fractional numbers.
Find a fractional part of a set of objects.	Multiply fractional numbers.
Write mathematical sentences for finding a fractional part of a set of objects.	Know and use 1 in the form $\frac{1}{1}, \frac{2}{2}, \frac{3}{3} \ldots$, as the identity element for multiplication of fractional numbers.
Use diagrams to show that multiplying or dividing the numerator and denominator of a fraction by the same number results in an equivalent fraction.	Know and use the commutative and associative properties for multiplication in the set of fractional numbers.
Find the product when the first factor is represented by a whole number and the second factor by a fractional number. Use repeated addition.	Know and use the distributive property of multiplication over addition on the set of fractional numbers.
Express the quotient of whole numbers as a fraction or a mixed numeral.	
Use ratios to represent situations such as 3 pencils for 10 cents as $\frac{3}{10}$.	Find the missing term in a proportion.
Determine if two ratios are equivalent in same manner as determining if two fractions are equivalent.	
Find the missing term in two given equivalent fractions or ratios using common denominators.	

SIXTH GRADE	SEVENTH GRADE
Add and subtract fractional numbers named by decimals.	
	Recognize that subtraction is always possible in the set of integers.
	Recognize that subtraction is always possible in the set of rational numbers.
	Recognize that all properties of fractional numbers also hold true for rational numbers.
Recognize the inverse relationship of multiplication and division of fractional numbers.	
	Explore the four operations in the set of integers and the set of rational numbers.
Recognize the reciprocal (multiplicative inverse) for every fractional number except zero and use it in division of fractional numbers.	
Recognize that division is always possible in the set of fractional numbers.	
Divide using set of fractional numbers.	
Multiply and divide decimal fractional numbers.	
Determine by division whether any fractional number has a repeating or terminating decimal equivalent.	Demonstrate understanding of fractions and decimals that name the same fractional number.
Interpret percent as a ratio in which the second number is always 100.	Treat percentage problems as proportions in which one of the equivalent ratios always has 100 as the denominator. $\frac{a}{b} = \frac{c}{100}$

Geometry

FOURTH GRADE	FIFTH GRADE
Recognize that a quadrilateral may have two pairs of parallel sides, one pair of parallel sides, or no pairs of parallel sides.	Recognize and name quadrilaterals.
Recognize that the 3 sides of a triangle may have the same measure, 2 sides may have the same measure, or all three have different measures.	Construct an isosceles triangle and an equilateral triangle.
	Construct the perpendicular bisector of a line segment.
Recognize without formal definition that a plane region is a closed path and the part of the plane which is inside the closed path.	Recognize that a plane is determined by three points not all on one line.
Know and use the concept of a circle as the set of all points in a plane that are the same distance from a fixed point on the interior. Know relationship between the length of radius and diameter.	Construct a circle given the radius.
	Recognize the possibilities of the intersection of a line and a plane (one point, no points—the empty set, line contained in the plane).
	Recognize parallel planes as planes which do not intersect.
	Recognize that when two planes intersect, the intersection is a line.
	Recognize that the intersection of a plane and a sphere is a circle, in some cases a great circle, or a single point.
Construct a line segment of the same length as a given line segment by use of compass and straightedge.	Know and use symbols for line, line segment, ray, angle, parallel lines, perpendicular lines, and congruence.
Recognize and name parallel lines as lines in a plane which do not intersect.	
Recognize a ray as starting at a point of origin and extending in one direction indefinitely.	
Recognize an angle as a figure formed by two rays having a common endpoint.	
Know, use, and construct a right angle.	
Use model of a right angle to construct perpendicular lines, right triangle, square, and rectangle.	

SIXTH GRADE	SEVENTH GRADE
Geometry	
Recognize pentagons, hexagons, octagons.	Recognize that two polygons having the same shape are similar and that their corresponding sides are proportional.
Recognize an ellipse.	Use informal methods to establish deductively the properties of geometric figures.
Recognize that an arc of a circle is a part of the circle.	
Recognize the subset relationship between sets of points, such as point to line, line to plane, and plane to space.	
Recognize that there is a point between any two points on a line.	
Know and use geometric symbols in writing simple mathematical sentences.	
Construct the bisector of an angle.	Construct an angle of the same measure as a given angle.
	Construct parallel lines.

FOURTH GRADE	FIFTH GRADE
Interpret, informally, space as the set of all points.	Recognize the characteristics of pyramids and cones.
Recognize the characteristics of spheres, rectangular prisms, and cylinders.	Recognize without formal definition that a solid region is a closed surface and the part of space which is inside the solid region.
Measurement	
	Use the metric system of linear measure.
Determine distance to the nearest quarter-inch and recognize the mile as a unit of distance.	Measure distance to nearest eighth-inch. Use odometer for measuring distances.
	Use measures of teaspoon and tablespoon.
Recognize the dry measure of pint, quart, peck, and bushel.	Use dry measures of pint, quart, peck, and bushel.
Recognize the relationships of standard units for liquid measures.	
Recognize the relationships of standard units for dry measures.	
Recognize freezing and boiling points.	
Recognize the second as a unit of time. Use decade and century as measures of time.	Recognize standard time zones.
Use ton to measure weight.	Recognize that other standard units of measure are used in special occupations.
Measure perimeter of quadrilaterals.	Find the perimeter of a polygon by addition.
	Use a formula to find the perimeter of regular triangles and quadrilaterals.
	Approximate the relationship between the circumference and diameter of a circle as 3.
	Recognize the degree as the standard unit of measure for angles.
	Recognize that a right angle has the measure of 90°
	Measure an angle by using a protractor.
	Recognize degree as a unit of measure for an arc of a circle.

SIXTH GRADE	SEVENTH GRADE
Measurement	
	Recognize and use metric units of measure for weight and volume.
Recognize the light year as a unit of measure for space distances.	
Estimate measurements of objects using appropriate units.	
	Measure temperature using the Celsius scale (centigrade).
Find perimeters of regular polygons by formula.	Demonstrate the Pythagorean theorem by using models of 3-4-5 and 5-12-13 right triangles.
	Find the circumference of a circle by formula.
Determine through paper folding or measurement that the sum of the measures of the angles of a triangle is 180°, a straight line.	

FOURTH GRADE	FIFTH GRADE
Measurement	
Recognize that a plane region is measured by comparing the plane region to a selected unit plane region to obtain a number. Find area of plane region by covering the plane region with chosen unit plane region.	Develop and use a formula to find the area of a rectangular region. Find surface areas of closed space regions.
	Recognize informally the concept of precision.
Recognize that a point in a plane (first quadrant) can be located by an ordered pair of numbers.	Use ordered pairs of numbers to locate points on a plane (first quadrant).
Read maps.	Understand scale drawings. Recognize latitude and longitude as a means of locating a point on the surface of a sphere, a globe. Perform the four arithmetic operations using numbers from measurements.

SIXTH GRADE	SEVENTH GRADE
Measurement	
Find the area of a circular region.	Compare areas of polygons having perimeters of the same measure.
	Approximate areas of irregular and curved plane regions by use of a grid.
Recognize that a space region is measured by comparing the space region to a selected unit space region to obtain a number.	
Find volume of space region by filling the space region.	Find the volume of a rectangular prism by experiment and by formula.
	Recognize that measures may be determined by direct and indirect measurement.
Determine which of two given measures is more precise.	Recognize that the greatest possible error of any measure is $\frac{1}{2}$ the unit used.
Find average (mean).	Find mean, median and mode of a set of numbers.
	Recognize that there is a point on a line for each integer.
	Recognize that there is a point on a line for each rational number.
	Locate points in a plane by an ordered pair of numbers (all four quadrants).
Make scale drawings.	
Use latitude and longitude to locate a point on the surface of sphere.	

SYMBOLS USED IN GUIDE

$+$	plus, add
$-$	minus, subtract
$\times$, $\cdot$	multiply
$\div$, $\overline{)\quad}$	divide
$\sqrt{}$	square root
$=$	is exactly the same as, is equal to
$\neq$	is not exactly the same as, is not equal to
$>$	is greater than
$<$	is less than
$\sim$	is similar to
$\approx$	is approximately the same as
$\cong$	is congruent to
$\ldots$	continued in this manner
Ø	empty set, null set
$\cup$	union
$\cap$	intersection
( )	parentheses
{ }	braces
[ ]	brackets
$.1\overline{6}$	bar, repeated digits
x^4	exponent
b_2	subscript
(a, b)	ordered pair
$\frac{a}{b}$	fraction, indicated division
12_{five}	base five
%	per cent
$^{+}5$	positive five
$^{-}5$	negative five
$\overline{AB}$	line segment, chord
$\overleftrightarrow{AB}$	line
$\overrightarrow{AB}$	ray
$\angle$	angle
$\odot$	circle
$\perp$	is perpendicular to
$\parallel$	is parallel to
$\widehat{AB}$	arc
π	pi, 3.1415+
$^\circ$	degree
″	inch, inches, second
′	foot, feet, minute
:	is to

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